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# Constructions for strictly cyclic 3-designs and applications to optimal OOCs with $\lambda = 2$ [☆]

Tao Feng [a], Yanxun Chang [a], Lijun Ji [b]

[a] *Institute of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China*
[b] *Department of Mathematics, Suzhou University, Suzhou 215006, PR China*

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ABSTRACT

In this paper we give some recursive constructions for strictly cyclic 3-designs. Using these constructions we have some infinite families of strictly cyclic Steiner quadruple systems and optimal optical orthogonal codes with weight 4 and index 2. As corollaries, many known constructions for strictly cyclic Steiner quadruple systems and optimal optical orthogonal codes are unified. We also notice that there does not exist an optimal $(n,4,2)$-OOC for any $n \equiv 0 \pmod{24}$. Thus we introduce the concept of strictly cyclic maximal packing quadruple systems to deal with the cases of $n \equiv 0 \pmod{24}$ for $(n,4,2)$-OOCs. By our recursive constructions, some infinite families are also given on strictly cyclic maximal packing quadruple systems.

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## 1. Introduction

Let $K$ be a set of positive integers. A *t-wise balanced design* ($t$BD) is a pair $(X, \mathcal{B})$, where $X$ is a set of $v$ points and $\mathcal{B}$ is a set of subsets of $X$ (called *blocks*), each of cardinality from $K$, such that every $t$-subset of $X$ is contained in a unique block. Such a design is denoted by $S(t, K, v)$. If $K = \{k\}$, we write $S(t, K, v)$ by $S(t, k, v)$ for brevity. An $S(3, 4, v)$ is called a *Steiner quadruple system* and denoted by SQS$(v)$, which is known to exist if and only if $v \equiv 2, 4 \pmod 6$ [11]. In what follows we always simply write $t$-design for $t$-wise balanced design.

An *automorphism group* of a $t$-design $(X, \mathcal{B})$ is a permutation group on $X$ leaving $\mathcal{B}$ invariant. A $t$-design is said to be *cyclic* if it admits an automorphism consisting of a cycle of length $v$. Without loss of generality we can identify $X$ with $Z_v$, the additive group of integers modulo $v$. If the stabilizer

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*E-mail addresses:* yxchang@bjtu.edu.cn (Y. Chang), jilijun@suda.edu.cn (L. Ji).

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of any block of a cyclic $t$-design is trivial, then the $t$-design is called *strictly cyclic*. A cyclic SQS($v$) is denoted by CSQS($v$) and a strictly cyclic SQS($v$) is denoted by sSQS($v$). It is easy to verify that an sSQS($v$) exists only if $v \equiv 2, 10 \pmod{24}$. According to the survey paper by Hartman and Phelps [14], the following partial results on CSQS($v$) are known.

**Theorem 1.1.** *(See [14].) There exists a* CSQS($v$) *for all* $v \equiv 2, 4 \pmod 6$ *with* $2 \leqslant v \leqslant 100$ *except for* $v \in \{8, 14, 16\}$ *and possibly for* $v \in \{46, 56, 62, 70, 86, 94\}$.

Much less is known about the existence of an sSQS($v$) despite of the efforts of many authors, for example, Köhler [20], Piotrowski [24] and Siemon [25–28]. For more information on the existence of SQS($v$) with other prescribed automorphism groups, the interested reader may refer to [10,14,21]. In this paper we only concentrate on 3-designs with the property of strictly cyclic.

The rest of this paper is organized as follows. In Section 2 we introduce the concept of strictly cyclic $s$-fan design to establish some recursive constructions for strictly cyclic $S(3, K, v)$. In Section 3 we present some constructions for semi-cyclic $H$ designs, which are used for input designs in the recursive constructions described in Section 2. In Section 4 the concept of cyclic 0-FG* is introduced. Then we give a recursive construction for cyclic 0-FG*s. Using this construction, together with the results in Sections 2 and 3, we obtain some infinite families of 0-FGs. In Section 5 a recursive construction and some infinite families of sSQS($v$) are shown. In Section 6 an application of our constructions from Sections 2–4 is given to optimal optical orthogonal codes of length $v$ with weight 4 and index 2. In Section 7 more new optimal $(v, 4, 2)$-OOCs are given by direct constructions.

## 2. Recursive constructions for strictly cyclic 3-designs

To describe our recursive constructions for strictly cyclic 3-designs, the following auxiliary designs are needed.

Let $s$ be a non-negative integer and $K_1, K_2, \dots, K_s$, $K_T$ be sets of positive integers. An *$s$-fan design* is an $(s+3)$-tuple $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_s, \mathcal{T})$ satisfying that $(X, \mathcal{G})$ is a 1-design, $(X, \mathcal{G} \cup \mathcal{B}_i)$ is a 2-design for each $1 \leqslant i \leqslant s$ and $(X, \mathcal{G} \cup (\bigcup_{i=1}^{s} \mathcal{B}_i) \cup \mathcal{T})$ is a 3-design. The elements of $\mathcal{G}$ and $(\bigcup_{i=1}^{s} \mathcal{B}_i) \cup \mathcal{T}$ are called *groups* and *blocks*, respectively. Suppose that there are $a_i$ groups of size $g_i$, $1 \leqslant i \leqslant m$. Then the type of an $s$-fan design is defined to be $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$. If block sizes of $\mathcal{B}_i$ and $\mathcal{T}$ are from $K_i$ $(1 \leqslant i \leqslant s)$ and $K_T$, respectively, then the $s$-fan design is denoted by $s$-FG$(3, (K_1, K_2, \dots, K_s, K_T), \sum_{i=1}^{m} g_i a_i)$ of type $g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$. Especially if the type is $1^v$, the $s$-fan design is simply written as $s$-fan $S(3, (K_1, K_2, \dots, K_s, K_T), v)$.

Hartman introduced the concept of $s$-fan design in [13]. Using this concept he gave an elegant construction for the existence of an SQS($v$). The strong power of this concept was shown further by Ji [15] and [16], who gave the existence of an $S(3, \{4, 5\}, v)$ and an $S(3, \{4, 5, 6\}, v)$. Actually there is an equivalent concept to $s$-fan design, called a candelabra 3-system. For more information on candelabra systems the interested reader may refer to [14] and [23].

For later use we need the necessary conditions for a 0-FG$(3, (\emptyset, K_T), gn)$ of type $g^n$ as follows.

**Lemma 2.1.** *The necessary conditions for the existence of a* 0-FG$(3, (\emptyset, K_T), gn)$ *of type* $g^n$ $(n \geqslant 2)$ *are*

(1) $g^2 n(n-1)(gn + g - 3) \equiv 0 \pmod{\alpha}$, *where* $\alpha = \gcd\{k(k-1)(k-2)\colon k \in K_T\}$;
(2) $g(n-1)(gn + g - 3) \equiv 0 \pmod{\beta}$, *where* $\beta = \gcd\{(k-1)(k-2)\colon k \in K_T\}$;
(3) *if* $g = 1$, *then* $n \equiv 2 \pmod{\gamma}$; *if* $g > 1$, *then* $gn \equiv g \equiv 2 \pmod{\gamma}$, *where* $\gamma = \gcd\{k - 2\colon k \in K_T\}$.

**Proof.** Counting the number of blocks in a 0-FG, (1) is obtained. Given $n$ distinct points $(n = 1, 2)$, counting the number of blocks that contain the $n$ points, (2) and (3) follow, respectively. Note that there are two different choices to fix 2 points. □

An *automorphism group* of an $s$-fan design $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ is a permutation group on $X$ leaving $\mathcal{G}$, $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s$, $\mathcal{T}$ invariant, respectively. Let $G$ be an automorphism group of an $s$-fan design. For any block $B$ of the $s$-fan design, the subgroup

$$\{\pi \in G\colon\ B^\pi = B\}$$

is called the *stabilizer* of $B$ in $G$. An $s$-fan design of type $g^n$ is said to be *cyclic* if it admits an automorphism consisting of a cycle of length $gn$. Without loss of generality we can identify $X$ with $Z_{gn}$ and $\mathcal{G}$ with $\{\{in + j\colon\ 0 \leqslant i \leqslant g-1\}\colon\ 0 \leqslant j \leqslant n-1\}$. If the stabilizer of any block of a cyclic $s$-fan design in $Z_{gn}$, the additive group of integers modulo $gn$, is trivial, then the $s$-fan design is called *strictly cyclic*.

Colbourn and Colbourn's definition [7] of $m$-beheaded cyclic SQS$(v)$ is a special case of our definition of cyclic $s$-fan design. An $m$-beheaded cyclic SQS$(v)$ is just a cyclic 0-FG$(3, (\emptyset, 4), v)$ of type $m^{v/m}$.

The following construction is simple but very useful.

**Construction 2.2.** *Suppose that there exists a cyclic* 0-FG$(3, (\emptyset, K_T), gn)$ *of type* $g^n$. *If there exists a cyclic* $S(3, L, g)$*, then there exists a cyclic* $S(3, K_T \cup L, gn)$*. Further if the master design* 0-FG *and the input design* $S(3, L, g)$ *are both strictly cyclic, then so is the resulting design.*

Construction 2.2 shows that for the purpose of constructing a strictly cyclic 3-design it is useful to find some strictly cyclic 0-FGs. More generally we will give the following recursive construction for strictly cyclic $s$-FGs, $s \geqslant 0$, which is a variation of Hartman's fundamental construction [13]. Before stating this construction we need the following concepts. For convenience, we always assume that $I_n = \{0, 1, \ldots, n-1\}$ where $n$ is a positive integer.

An $s$-fan design $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{T})$ of type $g^n$ is said to be *semi-cyclic* if it admits an automorphism consisting of $n$ cycles of length $g$. Without loss of generality we can identify $X$ with $I_n \times Z_g$ and $\mathcal{G}$ with $\{\{i\} \times Z_g\colon\ i \in I_n\}$. In this case the automorphism can be taken as $(i, x) + 1 \mapsto (i, x+1) \pmod{(-, g)}$, $i \in I_n$ and $x \in Z_g$. If the stabilizer of any block of a semi-cyclic $s$-fan design in $Z_g$ is trivial, i.e., for any block $B$, $\{\delta \in Z_g\colon\ B + \delta = B\} = \{0\}$, where $B + \delta = \{(i, x+\delta)\colon\ (i, x) \in B\}$, then the $s$-fan design is called *strictly semi-cyclic*.

Let $n$, $g$, $t$ be positive integers and $K$ be a set of positive integers. An *H design* is a triple $(X, \mathcal{G}, \mathcal{B})$, where $\mathcal{G}$ is a partition of a set of points $X$ into $n$ subsets (called *groups*), each of cardinality $g$, and $\mathcal{B}$ is a collection of subsets of $X$ (called *blocks*), each of size $k$, $k \in K$, such that each block intersects any given group in at most one point, and each $t$-subset of $X$ from $t$ distinct groups is contained in a unique block. Such a design is denoted by $H(n, g, K, t)$.

The early idea of an $H$ design can be found in Hanani [12], who used different terminology. Mills used the terminology $H$ design in [22]. In [13] Hartman gave a generalization of $H$ design, calling it tripartite design.

An *automorphism group* of an $H$ design $(X, \mathcal{G}, \mathcal{B})$ is a permutation group on $X$ leaving $\mathcal{G}$, $\mathcal{B}$ invariant, respectively. Let $G$ be an automorphism group of an $H$ design. For any block $B$ of the $H$ design, the subgroup

$$\{\pi \in G\colon\ B^\pi = B\}$$

is called the *stabilizer* of $B$ in $G$. An $H$ design of type $g^n$ is said to be *cyclic* if it admits an automorphism consisting of a cycle of length $gn$. Without loss of generality we can identify $X$ with $Z_{gn}$ and $\mathcal{G}$ with $\{\{in + j\colon\ 0 \leqslant i \leqslant g-1\}\colon\ 0 \leqslant j \leqslant n-1\}$. If the stabilizer of any block of a cyclic $H$ design in $Z_{gn}$ is trivial, then the $H$ design is called *strictly cyclic*.

If an $H$ design $(X, \mathcal{G}, \mathcal{B})$ of type $g^n$ admits an automorphism consisting of $n$ cycles of length $g$, then it is said to be *semi-cyclic*. We can always identify $X$ with $I_n \times Z_g$ and $\mathcal{G}$ with $\{\{i\} \times Z_g\colon$ $i \in I_n\}$. In this case the automorphism can be taken as $(i, x) + 1 \mapsto (i, x+1) \pmod{(-, g)}$, $i \in I_n$ and $x \in Z_g$. If the stabilizer of any block of a semi-cyclic $H$ design in $Z_g$ is trivial, i.e., for any block $B$, $\{\delta \in Z_g\colon\ B + \delta = B\} = \{0\}$, where $B + \delta = \{(i, x+\delta)\colon\ (i, x) \in B\}$, then the $H$ design is called *strictly semi-cyclic*. By the definition of $H$ design, one can verify that a semi-cyclic $H$ design is always strictly semi-cyclic. Otherwise, there exists at least one block whose stabilizer is not trivial. Let this block be $B = \{(i_j, x_j)\colon\ 1 \leqslant j \leqslant |B|\}$ where $i_j \in I_n$ and $x_j \in Z_g$ for $1 \leqslant j \leqslant |B|$. Then there exists an integer $\delta \in Z_g \setminus \{0\}$, such that $B + \delta = \{(i_j, x_j + \delta)\colon\ 1 \leqslant j \leqslant |B|\} = B$. Since $i_{j_1} \neq i_{j_2}$ for any

$1 \leqslant j_1 < j_2 \leqslant |B|$, we have that for any $1 \leqslant j \leqslant |B|$, $(i_j, x_j + \delta) = (i_j, x_j)$. Thus $x_j + \delta = x_j$. That is impossible for $\delta \in Z_g \setminus \{0\}$.

**Construction 2.3.** *Let $K$ and $L_i$ for each $1 \leqslant i \leqslant s$ be all sets of positive integers greater than 1. Let $K_T$ and $L_T$ be both sets of positive integers greater than 2. Suppose that there exists a strictly cyclic 1-FG$(3, (K, K_T), gn)$ of type $g^n$ (called a master design). If there exists a strictly semi-cyclic s-FG$(3, (L_1, L_2, \ldots, L_s, L_T), hk)$ of type $h^k$ for any $k \in K$, and a semi-cyclic $H(k, h, L_T, 3)$ for any $k \in K_T$, then there exists a strictly cyclic s-FG$(3, (L_1, L_2, \ldots, L_s, L_T), hgn)$ of type $(hg)^n$.*

**Proof.** Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ be a strictly cyclic 1-FG$(3, (K, K_T), gn)$ of type $g^n$ with $X = Z_{gn}$ and $\mathcal{G} = \{\{ni_1 + i_2\colon\ 0 \leqslant i_1 \leqslant g-1\}\colon\ 0 \leqslant i_2 \leqslant n-1\}$. Denote the family of base blocks of this design by $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ generate all the blocks of $\mathcal{B}$ and $\mathcal{T}$, respectively.

For any base block $B \in \mathcal{F}_1$, construct a strictly semi-cyclic $s$-FG$(3, (L_1, L_2, \ldots, L_s, L_T), h|B|)$ of type $h^{|B|}$ on $B \times Z_h$ with groups $\{\{x\} \times Z_h\colon\ x \in B\}$. Denote the family of base blocks of the $j$th subdesign $GDD(2, L_j, h|B|)$ by $\mathcal{A}_B^j$ for $1 \leqslant j \leqslant s$, and denote the family of all the other base blocks by $\mathcal{D}_B$. Let $\mathcal{A}_B = \bigcup_{j=1}^{s} \mathcal{A}_B^j$.

For any base block $B \in \mathcal{F}_2$, construct a semi-cyclic $H(|B|, h, L_T, 3)$ on $B \times Z_h$ with groups $\{\{x\} \times Z_h\colon\ x \in B\}$. Denote the family of base blocks of this design by $\mathcal{D}'_B$.

Let $X' = Z_{hgn}$ and $\mathcal{G}' = \{\{ni'_1 + i'_2\colon\ 0 \leqslant i'_1 \leqslant hg-1\}\colon\ 0 \leqslant i'_2 \leqslant n-1\}$. Let $\mathcal{A}_j = \bigcup_{B \in \mathcal{F}_1} \mathcal{A}_B^j$ for $1 \leqslant j \leqslant s$ and $\mathcal{D} = (\bigcup_{B \in \mathcal{F}_1} \mathcal{D}_B) \cup (\bigcup_{B \in \mathcal{F}_2} \mathcal{D}'_B)$. Now we construct a strictly cyclic $s$-FG$(3, (L_1, L_2, \ldots, L_s, L_T), hgn)$ of type $(hg)^n$ as follows: For each $C \in (\bigcup_{1 \leqslant j \leqslant s} \mathcal{A}_j) \cup \mathcal{D}$ and each $(x, y) \in C$, define a mapping

$$\tau : (x, y) \mapsto x + ygn \pmod{hgn}.$$

Define $\tau(C) = \{\tau(c)\colon\ c \in C\}$. Let

$$\mathcal{A}_j^* = \bigcup_{C \in \mathcal{A}_j} \tau(C), \quad 1 \leqslant j \leqslant s, \qquad \mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C).$$

Let

$$\mathcal{A}'_j = \{A + \delta\colon\ A \in \mathcal{A}_j^*,\ \delta \in Z_{hgn}\}, \qquad \mathcal{D}' = \{A + \delta\colon\ A \in \mathcal{D}^*,\ \delta \in Z_{hgn}\},$$

where $A + \delta = \{z + \delta \pmod{hgn}\colon\ z \in A\}$. Then $(X', \mathcal{G}', \mathcal{A}'_1, \ldots, \mathcal{A}'_s, \mathcal{D}')$ is the required design.

For checking the required design, it suffices to show that: (1) the resulting design is strictly cyclic; (2) any 3-subset $S$, $S \subset X'$, $|S \cap G'| < 3$ for all $G' \in \mathcal{G}'$, is contained in a unique block of the resulting design; (3) any 2-subset $R$, $R \subset X'$, $|R \cap G'| < 2$ for all $G' \in \mathcal{G}'$, is contained in a unique block of $\mathcal{A}'_i$ for each $1 \leqslant i \leqslant s$.

(1) Suppose that $A = \{x_1 + y_1 gn, x_2 + y_2 gn, \ldots, x_r + y_r gn\}$ is a base block of the resulting design, where $0 \leqslant x_l \leqslant gn-1$, $0 \leqslant y_l \leqslant h-1$, $1 \leqslant l \leqslant r$. We need to show that the stabilizer of $A$ is trivial, i.e., $A + \delta = A$ if and only if $\delta \equiv 0 \pmod{hgn}$. The sufficiency follows immediately, so we consider the necessity.

Assume that $\delta = \delta_1 + \delta_2 gn$, $0 \leqslant \delta_1 \leqslant gn-1$, $0 \leqslant \delta_2 \leqslant h-1$. If $A + \delta = A$, we have

$$\{x_l + y_l gn\colon\ 1 \leqslant l \leqslant r\} = \{x_l + \delta_1 + (y_l + \delta_2)gn\colon\ 1 \leqslant l \leqslant r\},$$

where the arithmetic is modulo $hgn$. It follows that

$$\{x_l\colon\ 1 \leqslant l \leqslant r\} = \{x_l + \delta_1\colon\ 1 \leqslant l \leqslant r\},$$

where the arithmetic is modulo $gn$. Let $U = \{x_l\colon\ 1 \leqslant l \leqslant r\}$.

If $A \in \mathcal{A}'_j$, $1 \leqslant j \leqslant s$, then $|U| = r \geqslant 2$. Since the subdesign $(X, \mathcal{G}, \mathcal{B})$ of the master design 1-FG$(3, (K, K_T), gn)$ of type $g^n$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{T})$ is strictly cyclic and it requires that any 2-subset of $X$ which intersects any group of $\mathcal{G}$ in at most one point occurs in exactly one block, we have $\delta_1 = 0$.

If $A \in \mathcal{D}'$, without loss of generality we can always assume that $A \in \mathcal{D}^*$. If $A = \tau(C)$ for some $C \in \bigcup_{B \in \mathcal{F}_2} \mathcal{D}'_B$, then $|U| = r \geqslant 3$. Since the master design 1-FG$(3, (K, K_T), gn)$ of type $g^n$ is strictly cyclic and it requires that any 3-subset of $X$ which intersects any group of $\mathcal{G}$ in at most two points occurs in exactly one block, we have $\delta_1 = 0$. If $A = \tau(C)$ for some $C \in \bigcup_{B\in\mathcal{F}_1} \mathcal{D}_B$, then $|U| \geqslant 2$. Note that in this case $U$ may be a multiset, i.e., $|U|$ may be not equal to $r$. By similar arguments as the case of $A \in \mathcal{A}'_j$, we have $\delta_1 = 0$.

Hence,

$$\{x_l + y_l gn\colon\ 1 \leqslant l \leqslant r\} = \{x_l + (y_l + \delta_2)gn\colon\ 1 \leqslant l \leqslant r\},$$

where the arithmetic is modulo $hgn$. Since the input designs are all strictly semi-cyclic, we have $\delta_2 = 0$. Thus the resulting design is strictly cyclic.

(2) Take any triple $S = \{z_1, z_2, z_3\} \subset X'$, where $z_1, z_2, z_3$ are not equal modulo $n$ at the same time. Let $z_l = x_l + y_l gn$, $0 \leqslant x_l \leqslant gn - 1$, $0 \leqslant y_l \leqslant h - 1$, $1 \leqslant l \leqslant 3$. We consider the following cases.

*Case* 1. Suppose that $x_1$, $x_2$, $x_3$ are pairwise distinct modulo $n$. Then there exists a unique base block $F$ in $\mathcal{F}$ and a unique element $\delta_1 \in Z_{gn}$, such that $\{x_1^*, x_2^*, x_3^*\} \subseteq F$ and $x_l^* + \delta_1 = x_l + \sigma_l gn$ for some $\sigma_l \in \{0, 1\}$, $1 \leqslant l \leqslant 3$. It follows that $z_l - \delta_1 = x_l^* + (y_l - \sigma_l)gn$. Note that $x_1^*$, $x_2^*$, $x_3^*$ are also pairwise distinct modulo $n$.

If $F \in \mathcal{F}_1$, then there exists a unique base block $C \in \mathcal{A}_F \cup \mathcal{D}_F$ and a unique element $\delta_2 \in Z_h$, such that $\{(x_1^*, y_1^*), (x_2^*, y_2^*), (x_3^*, y_3^*)\} \subseteq C$ and $(x_l^*, y_l^* + \delta_2) = (x_l^*, y_l - \sigma_l + \sigma_l' h)$ for some $\sigma_l' \in \{0, 1\}$, $1 \leqslant l \leqslant 3$. By the mapping $\tau$, we have that $x_l^* + (y_l^* + \delta_2)gn = x_l^* + (y_l - \sigma_l + \sigma_l' h)gn = z_l - \delta_1 + \sigma_l' hgn$. Let $\delta = \delta_1 + \delta_2 gn$. It follows that $z_l = x_l^* + y_l^* gn + \delta$, where the arithmetic is modulo $hgn$. By (1) the resulting design is strictly cyclic, so $\{z_1, z_2, z_3\}$ is contained in the unique block $\tau(C) + \delta$, which is generated by $\tau(C)$. Similar arguments hold for $F \in \mathcal{F}_2$, where $C \in \mathcal{D}'_F$.

*Case* 2. Suppose that $x_1 \equiv x_2 \pmod n$, $x_1 \neq x_2$, and $x_1 \not\equiv x_3 \pmod n$. By similar arguments as in Case 1, there exists a unique base block $F \in \mathcal{F}_2$, a unique base block $C \in \mathcal{D}'_F$, a unique element $\delta_1 \in Z_{gn}$ and a unique element $\delta_2 \in Z_h$, such that $\{z_1, z_2, z_3\}$ is contained in the unique block $\tau(C) + \delta$, where $\delta = \delta_1 + \delta_2 gn$, which is generated by $\tau(C)$.

*Case* 3. Suppose that $x_1 = x_2$, $y_1 \neq y_2$ and $x_1 \not\equiv x_3 \pmod n$. By similar arguments as in Case 1, there exists a unique base block $F \in \mathcal{F}_1$, a unique base block $C \in \mathcal{D}_F$, a unique element $\delta_1 \in Z_{gn}$ and a unique element $\delta_2 \in Z_h$, such that $\{z_1, z_2, z_3\}$ is contained in the unique block $\tau(C) + \delta$, where $\delta = \delta_1 + \delta_2 gn$, which is generated by $\tau(C)$.

(3) Take any 2-subset $R = \{z_1, z_2\}$, where $z_l = x_l + y_l gn$, $0 \leqslant x_l \leqslant gn - 1$, $0 \leqslant y_l \leqslant h - 1$, $1 \leqslant l \leqslant 2$ and $x_1$, $x_2$ are distinct modulo $n$. Then there exists a unique base block $F$ in $\mathcal{F}_1$ and a unique element $\delta_1 \in Z_{gn}$, such that $\{x_1^*, x_2^*\} \subseteq F$ and $x_l^* + \delta_1 = x_l + \sigma_l gn$ for some $\sigma_l \in \{0, 1\}$, $1 \leqslant l \leqslant 2$. It follows that $z_l - \delta_1 = x_l^* + (y_l - \sigma_l)gn$. Note that $x_1^*$, $x_2^*$ are also distinct modulo $n$.

Then, given any $1 \leqslant j \leqslant s$, there exists a unique base block $C_j$ in $\mathcal{A}_j^*$ and a unique element $\delta_{2j} \in Z_h$, such that $\{(x_1^*, y_{1j}^*), (x_2^*, y_{2j}^*)\} \subseteq C_j$ and $(x_l^*, y_{lj}^* + \delta_{2j}) = (x_l^*, y_l - \sigma_l + \sigma_{lj}' h)$ for some $\sigma_{lj}' \in \{0, 1\}$, $1 \leqslant l \leqslant 2$. By the mapping $\tau$, we have that $x_l^* + (y_{lj}^* + \delta_{2j})gn = x_l^* + (y_l - \sigma_l + \sigma_{lj}' h)gn = z_l - \delta_1 + \sigma_{lj}' hgn$. Let $\delta = \delta_1 + \delta_{2j} gn$. It follows that $z_l = x_l^* + y_{lj}^* gn + \delta$, where the arithmetic is modulo $hgn$. By (1) the resulting design is strictly cyclic, so $\{z_1, z_2\}$ is contained in the unique block $\tau(C_j) + \delta$, which is generated by $\tau(C_j)$.

This completes the proof. $\square$

In the case of $K = \{2\}$ in Construction 2.3, we have the following corollary to construct a strictly cyclic 0-FG from a strictly cyclic 0-FG.

**Construction 2.4.** *Let $g \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 2$. Suppose that there exists a strictly cyclic 0-FG$(3, (\emptyset, K_T), gn)$ of type $g^n$. If there exists a semi-cyclic $H(k, h, L_T, 3)$ for any $k \in K_T$ and a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2h)$ of type $h^2$, then there exists a strictly cyclic 0-FG$(3, (\emptyset, L_T \cup \{4\}), hgn)$ of type $(hg)^n$.*

**Proof.** If we can prove that there exists a strictly cyclic 1-FG$(3, (2, K_T), gn)$ of type $g^n$, then applying Construction 2.3 with a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2h)$ of type $h^2$ and a semi-cyclic $H(k, h, L_T, 3)$ for each $k \in K_T$ gives the conclusion.

Let $(X, \mathcal{G}, \emptyset, \mathcal{T})$ be a strictly cyclic 0-FG$(3, (\emptyset, K_T), gn)$ of type $g^n$. Let

$$\mathcal{R} = \begin{cases} \{\{0, j\}:\ 0 \leqslant j \leqslant \frac{gn}{2} - 1,\ n \nmid j\}, & \text{if } g \equiv 0 \pmod 2, \\ \{\{0, j\}:\ 0 \leqslant j \leqslant \frac{gn-1}{2},\ n \nmid j\}, & \text{if } gn \equiv 1 \pmod 2. \end{cases}$$

Obviously the development of $\mathcal{R}$, $\{R + l:\ R \in \mathcal{R},\ l \in Z_{gn}\}$, includes each 2-subset of $X$ exactly once which intersects any group of $\mathcal{G}$ in at most one point, where $R + l = \{r + l \pmod{gn}:\ r \in R\}$. Furthermore, due to $g \equiv 0 \pmod 2$ or $gn \equiv 1 \pmod 2$, if $R + l = R$, i.e., $\{l, j + l\} = \{0, j\}$, where the arithmetic is modulo $gn$, then we have $l = 0$.

It is readily checked that $(X, \mathcal{G}, \mathcal{R}, \mathcal{T})$ is a strictly cyclic 1-FG$(3, (2, K_T), gn)$ of type $g^n$. This completes the proof. □

Note that by Lemma 2.1(3), if $K_T = \{4\}$ and $g \neq 1$, $g$ is always congruent to 0 modulo 2. The following construction shows that how to construct a strictly cyclic 0-FG from a strictly cyclic $H$ design.

**Construction 2.5.** *Suppose that there exists a strictly cyclic $H(n, g, K, 3)$. If there exists a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2g)$ of type $g^2$, then there exists a strictly cyclic 0-FG$(3, (\emptyset, K \cup \{4\}), gn)$ of type $h^m$, where $(h, m) = (g, n)$ when $n \equiv 1 \pmod 2$ and $(h, m) = (2g, n/2)$ when $n \equiv 0 \pmod 2$.*

**Proof.** Let $(X, \mathcal{G}, \mathcal{B})$ be a strictly cyclic $H(n, g, K, 3)$ with $X = Z_{gn}$ and $\mathcal{G} = \{\{ni_1 + i_2:\ 0 \leqslant i_1 \leqslant g - 1\}:\ 0 \leqslant i_2 \leqslant n - 1\}$. Denote the family of base blocks of this design by $\mathcal{F}$.

Let

$$\mathcal{R} = \begin{cases} \{\{0, j\}:\ 1 \leqslant j \leqslant \frac{n-1}{2}\}, & \text{if } n \equiv 1 \pmod 2, \\ \{\{0, j\}:\ 1 \leqslant j \leqslant \frac{n}{2} - 1\}, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

For any $R \in \mathcal{R}$, construct a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2g)$ of type $g^2$ on $R \times Z_g$ with groups $\{\{x\} \times Z_g:\ x \in R\}$. Denote the family of base blocks of this design by $\mathcal{D}_R$.

Let $\mathcal{D} = \bigcup_{R \in \mathcal{R}} \mathcal{D}_R$. For each $C \in \mathcal{D}$ and each $(x, y) \in C$, define a mapping

$$\tau : (x, y) \mapsto x + yn \pmod{gn}.$$

Define $\tau(C) = \{\tau(c):\ c \in C\}$. Let

$$\mathcal{D}^* = \bigcup_{C \in \mathcal{D}} \tau(C), \qquad \mathcal{D}' = \{D + \delta:\ D \in \mathcal{D}^*,\ \delta \in Z_{gn}\},$$

where $D + \delta = \{z + \delta \pmod{gn}:\ z \in D\}$. Let $\mathcal{G}' = \{\{mi_1 + i_2:\ 0 \leqslant i_1 \leqslant h - 1\}:\ 0 \leqslant i_2 \leqslant m - 1\}$. Then $(X, \mathcal{G}', \emptyset, \mathcal{B} \cup \mathcal{D}')$ is the required design.

For checking the required design, it suffices to show that: (1) the stabilizer of any block from $\mathcal{D}'$ is trivial; (2) any 3-subset $S$, $S \subset X$, $|S \cap G| = 2$ for some $G \in \mathcal{G}'$, is contained in a unique block of the resulting design.

Note that if $n \equiv 1 \pmod 2$, the development of $\mathcal{R}$, $\{R + l:\ R \in \mathcal{R},\ l \in Z_n\}$, includes each 2-subset of $Z_n$ exactly once, where $R + l = \{r + l \pmod n:\ r \in R\}$. If $n \equiv 0 \pmod 2$, the development of $\mathcal{R}$ includes each 2-subset of $Z_n$ whose differences of two elements are not equal to $n/2$ exactly once. Furthermore, if $R + l = R$, i.e., $\{l, j + l\} = \{0, j\}$, where the arithmetic is modulo $n$, then we have $l = 0$ for any positive integer $n$.

Similar arguments as in Construction 2.3, it is readily checked that (1) and (2) hold. This completes the proof. □

Construction 2.5 is analogous to Lemma 4.4 in [4], which used a different tool called *a matched factor system* to establish a similar result. Note that the necessary conditions for the existence of a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2g)$ of type $g^2$ and a matched factor system of order $g$ is $g \equiv 0 \pmod 4$ and $g \equiv 0 \pmod 8$, respectively, so Construction 2.5 is a possible generalization of Lemma 4.4 in [4]. On the existence of a strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 2g)$ of type $g^2$, we will give some results in Section 4.

## 3. Constructions for semi-cyclic $H(n, g, K, t)$

To make use of Constructions 2.3–2.5, we need to construct some semi-cyclic or strictly cyclic $H(n, g, K, t)$'s. First we give an equivalent relation between a semi-cyclic $H$ design and an $r$-simple matrix as follows.

Let $G$ be an abelian group of order $g$ and $r$ be a positive integer. An $s \times t$ matrix $A = (a_{ij})$ over $G$ is $r$-simple if the difference of any two column vectors of $A$ contains each element of $G$ at most $r - 1$ times.

**Theorem 3.1.** *A semi-cyclic $H(k, g, k, t)$ is equivalent to a $k \times g^{t-1}$ $t$-simple matrix over $Z_g$.*

**Proof.** Suppose that there exists a semi-cyclic $H(k, g, k, t)$ on $I_k \times Z_g$ with groups $\{\{x\} \times Z_g\colon x \in I_k\}$. Simply counting shows that the number of base blocks of this design is $g^{t-1}$. Denote its base blocks by $B_1, B_2, \ldots, B_{g^{t-1}}$. Now define a $k \times g^{t-1}$ matrix $D = (d_{ij})$, $i \in I_k$, $1 \leqslant j \leqslant g^{t-1}$, as follows:

$$d_{ij} = x \quad \Longleftrightarrow \quad (i, x) \in B_j.$$

It is readily checked that the resulting matrix is a $k \times g^{t-1}$ $t$-simple matrix. Conversely, given a $k \times g^{t-1}$ $t$-simple matrix over $Z_g$, we can form a semi-cyclic $H(k, g, k, t)$ by reversing the above process. The assertion then follows. □

**Lemma 3.2.** *(See [5].) For any prime $p$ and any integer $t$ with $2 \leqslant t \leqslant p$, there exists a $p \times p^{t-1}$ $t$-simple matrix over $Z_p$.*

**Lemma 3.3.** *(See [3].) For any $g \geqslant 2$, there exists a $4 \times g^2$ 3-simple matrix over $Z_g$.*

**Corollary 3.4.** *For any prime $p$ and an integer $t$ with $2 \leqslant t \leqslant p$, there exists a semi-cyclic $H(p, p, p, t)$.*

**Corollary 3.5.** *For any $g \geqslant 1$, there exists a semi-cyclic $H(4, g, 4, 3)$.*

By Corollary 3.5 we have a more general result as follows.

**Lemma 3.6.** *For any $g \geqslant 1$, $n \geqslant 4$ and $n \equiv 2, 4 \pmod 6$, there exists a semi-cyclic $H(n, g, 4, 3)$.*

**Proof.** As pointed out in Section 1, it is known that there exists an SQS$(n)$ for any $n \equiv 2, 4 \pmod 6$ [11]. Let $(X, \mathcal{B})$ be an SQS$(n)$ with $X = I_n$. For any $B \in \mathcal{B}$, by Corollary 3.5 we construct a semi-cyclic $H(4, g, 4, 3)$ on $B \times Z_g$ with groups $\{\{x\} \times Z_g\colon x \in B\}$ and block-set $\mathcal{A}_B$. Let $X' = I_n \times Z_g$ and $\mathcal{G}' = \{\{x\} \times Z_g\colon x \in I_n\}$. Let $\mathcal{B}' = \bigcup_{B \in \mathcal{B}} \mathcal{A}_B$. Then it is readily checked that $(X', \mathcal{G}', \mathcal{B}')$ is the required design. □

**Lemma 3.7.** *If there exists a semi-cyclic $H(k, g, k, t)$, then there exists a semi-cyclic $H(r, g, r, t)$ for any $t \leqslant r \leqslant k$.*

**Proof.** By the definition of semi-cyclic $H$-design, the verification is straightforward. □

Combining Corollary 3.4 and Lemma 3.7, we have

**Corollary 3.8.** *Let $p$ be a prime. For any integer $t$ with $2 \leqslant t \leqslant p$ and any integer $r$ with $t \leqslant r \leqslant p$, there exists a semi-cyclic $H(r, p, r, t)$.*

In the following a recursive construction for strictly cyclic $H(n, g, K, t)$ is given. Note that it is easy to verify that a strictly cyclic $H(n, g, K, t)$ is also a semi-cyclic $H(n, g, K, t)$.

**Construction 3.9.** *Suppose that there exists a strictly cyclic $H(n,g,K,t)$. If there exists a semi-cyclic $H(k,h,L,t)$ for any $k \in K$, then there exists a strictly cyclic $H(n,gh,L,t)$.*

**Proof.** Let $(X,\mathcal{G},\mathcal{B})$ be a strictly cyclic $H(n,g,K,t)$ with $X = Z_{gn}$ and $\mathcal{G} = \{\{ni_1 + i_2\colon\ 0 \leqslant i_1 \leqslant g-1\}\colon\ 0 \leqslant i_2 \leqslant n-1\}$. Denote the family of base blocks of this design by $\mathcal{F}$.

For any base block $B \in \mathcal{F}$, construct a semi-cyclic $H(|B|,h,L,t)$ on $B \times Z_h$ with groups $\{\{x\} \times Z_h\colon x \in B\}$. Denote the family of base blocks of this design by $\mathcal{D}_B$.

Let $X' = Z_{hgn}$ and $\mathcal{G}' = \{\{ni'_1 + i'_2\colon\ 0 \leqslant i'_1 \leqslant hg-1\}\colon\ 0 \leqslant i'_2 \leqslant n-1\}$. Let $\mathcal{D} = \bigcup_{B\in\mathcal{F}} \mathcal{D}_B$. Now we construct a strictly cyclic $H(n,gh,L,t)$ as follows: For each $C \in \mathcal{D}$ and each pair $(x,y) \in C$, define a mapping

$$\tau : (x,y) \mapsto x + ygn \pmod{hgn}.$$

Define $\tau(C) = \{\tau(c)\colon\ c \in C\}$. Let

$$\mathcal{D}^* = \bigcup_{C\in\mathcal{D}} \tau(C), \qquad \mathcal{D}' = \{D + \delta\colon\ D \in \mathcal{D}^*,\ \delta \in Z_{hgn}\},$$

where $D + \delta = \{z + \delta \pmod{hgn}\colon\ z \in D\}$. Then similar arguments as in Construction 2.3, it is readily checked that $(X',\mathcal{G}',\mathcal{D}')$ is the required design. $\square$

Combining Corollary 3.5 and Construction 3.9, we have the following:

**Corollary 3.10.** *Suppose that there exists a strictly cyclic $H(n,g,4,3)$. Then there exists a strictly cyclic $H(n,gh,4,3)$ for any $h \geqslant 1$.*

**Corollary 3.11.** *(See [4].) Suppose that there exists an* sSQS$(n)$. *Then there exists a strictly cyclic $H(n,g,4,3)$ for any $g \geqslant 1$.*

Construction 3.9 is a generalization of Theorem 3.2 in [4] for strictly cyclic $H$ design, which deals with the case of $g = 1$.

**Lemma 3.12.** *(See [4].) If there exists a* CSQS$(n)$ *for $n \equiv 2, 10 \pmod{12}$, then for any $g \geqslant 2$ and $g \equiv 0 \pmod 2$, there exists a strictly cyclic $H(n,g,4,3)$.*

## 4. Constructions for strictly cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$

In this section we will present a recursive construction for strictly cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$. Combining this construction and the results in Sections 2 and 3 some infinite families of strictly cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$ are given.

By Lemma 2.1(3), if there exists a cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$ and $g \neq 1$, then $gn \equiv g \equiv 0 \pmod 2$. Then for $1 \leqslant i \leqslant gn/2$ and $i \not\equiv 0 \pmod n$, there exist $g(n-1)/2$ triples of form $\{0,i,2i\}$ and $\{0,i,gn/2\}$, respectively, in a cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$. Therefore we introduce the following conception.

Let $gn \equiv g \pmod 4$. In a cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$, if any triple of form $\{j,j+i,j+2i\}$ or $\{j,j+i,j+gn/2\}$, where $1 \leqslant i \leqslant gn/2$, $i \not\equiv 0 \pmod n$ and $0 \leqslant j \leqslant gn-1$, is contained in the block $\{j,j+a,j+2a,j+\frac{gn}{2}+a\}$ for some $1 \leqslant a \leqslant \lfloor\frac{gn}{4}\rfloor$ and $a \not\equiv 0 \pmod n$, then such a cyclic 0-FG is denoted by cyclic 0-FG*.

Note that for a given $j$, there are $g(n-1)/4$ blocks of form $\{j,j+a,j+2a,j+\frac{gn}{2}+a\}$ in a cyclic 0-FG*$(3,(\emptyset,4),gn)$ of type $g^n$. Thus we need the condition $gn \equiv g \pmod 4$ in the definition of 0-FG*. It is easy to see that a cyclic 0-FG*$(3,(\emptyset,4),gn)$ of type $g^n$ contains no half-orbit. If it contains a quarter-orbit, then $gn \equiv 0 \pmod 4$. It follows that $g \equiv 0 \pmod 4$ and each block of the quarter-orbit is contained in one of groups of the 0-FG*. That is impossible. Therefore a cyclic 0-FG*$(3,(\emptyset,4),gn)$ of type $g^n$ is always strictly cyclic.

**Construction 4.1.** *Suppose that $h \geqslant 3$ be an odd integer. If there exists a cyclic 0-FG*$^*(3,(\emptyset,4),gn)$ *of type* $g^n$, *then there exists a cyclic 0-FG*$^*(3,(\emptyset,4),hgn)$ *of type* $(hg)^n$.

**Proof.** Let $(Z_{gn},\mathcal{G},\emptyset,\mathcal{T})$ be a cyclic 0-FG$^*(3,(\emptyset,4),gn)$ of type $g^n$ with $\mathcal{G}=\{\{ni_1+i_2\colon 0\leqslant i_1\leqslant g-1\}\colon 0\leqslant i_2\leqslant n-1\}$. Denote the family of base blocks of this design by $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$, where $\mathcal{F}_1$ consists of all base blocks in the form of $\{0,a,2a,\frac{gn}{2}+a\}$ for $1\leqslant a\leqslant\lfloor\frac{gn}{4}\rfloor$, $a\not\equiv 0 \pmod n$, and $\mathcal{F}_2$ consists of all the other base blocks. It is easy to see that $|\mathcal{F}_1|=g(n-1)/4$ and $|\mathcal{F}_2|=|\mathcal{F}|-|\mathcal{F}_1|=g(n-1)(gn+g-9)/24$.

We shall construct the required cyclic 0-FG$^*(3,(\emptyset,4),hgn)$ of type $(hg)^n$ on $Z_{hgn}$ with groups $\mathcal{G}'=\{\{ni'_1+i'_2\colon 0\leqslant i'_1\leqslant hg-1\}\colon 0\leqslant i'_2\leqslant n-1\}$.

(i) For any base block $B\in\mathcal{F}_2$, construct a semi-cyclic $H(4,h,4,3)$ on $B\times Z_h$ with groups $\{\{x\}\times Z_h\colon x\in B\}$. Such a design exists by Corollary 3.5. Denote the family of base blocks of this design by $\mathcal{A}_B$. For each $A=\{(x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4)\}\in\mathcal{A}_B$, let $A'=\{x_1+y_1gn,x_2+y_2gn,x_3+y_3gn,x_4+y_4gn\}$. Denote $\mathcal{C}_B=\{A'\colon A\in\mathcal{A}_B\}$ and $\mathcal{C}_1=\bigcup_{B\in\mathcal{F}_2}\mathcal{C}_B$.

(ii) Let $\mathcal{C}_2=\{\{0,i,2i,\frac{hgn}{2}+i\}\colon 1\leqslant i\leqslant\lfloor\frac{hgn}{4}\rfloor,\ i\not\equiv 0 \pmod n\}$.

(iii) Let $\mathcal{C}_3=\{\{0,i,jgn+i,jgn+2i\}\colon 1\leqslant j\leqslant h-1,\ 1\leqslant i\leqslant (h-j)gn/2,\ i\not\equiv 0 \pmod n\}$.

(iv) Let $\mathcal{C}_4=\{\{0,jgn+i,kgn+\frac{gn}{2},(k+j)gn+\frac{gn}{2}+i\}\colon 0\leqslant j\leqslant (h-1)/2,\ 0\leqslant k\leqslant (h-3)/2,\ 1\leqslant i\leqslant\lfloor\frac{gn}{4}\rfloor,\ i\not\equiv 0 \pmod n\}$.

(v) Let $\mathcal{C}_5=\{\{0,jgn+\frac{gn}{2}+\lfloor\frac{gn}{4}\rfloor+i,kgn+\frac{gn}{2},(k+j+1)gn+\lfloor\frac{gn}{4}\rfloor+i\}\colon 0\leqslant j,k\leqslant (h-3)/2,\ 1\leqslant i\leqslant\lfloor\frac{gn}{4}\rfloor,\ i\not\equiv 0 \pmod n\}$.

Let $\mathcal{C}'_i=\{C+j\colon C\in\mathcal{C}_i,\ j\in Z_{hgn}\}$ for $1\leqslant i\leqslant 5$. Let $\mathcal{T}'=\bigcup_{i=1}^5\mathcal{C}'_i$. Then $(Z_{hgn},\mathcal{G}',\emptyset,\mathcal{T}')$ is the required 0-FG$^*(3,(\emptyset,4),hgn)$ of type $(hg)^n$.

For checking the resulting design, first we observe the triples covered by $\mathcal{C}'_4$ and $\mathcal{C}'_5$. Let $I=\{i\colon 1\leqslant i\leqslant\lfloor\frac{gn}{4}\rfloor,\ i\not\equiv 0 \pmod n\}$. For $0\leqslant k\leqslant h-1$, define

$$S_1(k)=\left\{\left\{0,jgn+i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-1)/2,\ i\in I\right\},$$

$$S_2(k)=\left\{\left\{0,-(jgn+i),kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-1)/2,\ i\in I\right\},$$

$$S_3(k)=\left\{\left\{0,(k+j)gn+\frac{gn}{2}+i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-1)/2,\ i\in I\right\},$$

$$S_4(k)=\left\{\left\{0,(k-j)gn+\frac{gn}{2}-i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-1)/2,\ i\in I\right\},$$

$$S_5(k)=\left\{\left\{0,jgn+\frac{gn}{2}+\left\lfloor\frac{gn}{4}\right\rfloor+i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-3)/2,\ i\in I\right\},$$

$$S_6(k)=\left\{\left\{0,-\left(jgn+\frac{gn}{2}+\left\lfloor\frac{gn}{4}\right\rfloor+i\right),kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-3)/2,\ i\in I\right\},$$

$$S_7(k)=\left\{\left\{0,(k+j)gn+gn+\left\lfloor\frac{gn}{4}\right\rfloor+i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-3)/2,\ i\in I\right\},$$

$$S_8(k)=\left\{\left\{0,(k-j)gn-\left\lfloor\frac{gn}{4}\right\rfloor-i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant (h-3)/2,\ i\in I\right\}.$$

Let $S'_r(k)=\{\{s+t\colon s\in S_r(k)\}\colon 0\leqslant t\leqslant hgn-1\}$ for $1\leqslant r\leqslant 8$. It is easy to see that the union of triples covered by $\mathcal{C}'_4$ are $\bigcup_{k=0}^{(h-3)/2}\bigcup_{r=1}^4 S'_r(k)$ and the union of triples covered by $\mathcal{C}'_5$ are $\bigcup_{k=0}^{(h-3)/2}\bigcup_{r=5}^8 S'_r(k)$. For a given $0\leqslant k\leqslant (h-3)/2$, it is readily checked that

$$S_1(k)\cup S_6(k)=\left\{\left\{0,jgn+i,kgn+\frac{gn}{2}\right\}\colon 0\leqslant j\leqslant h-1,\ 1\leqslant i\leqslant\left\lfloor\frac{gn}{4}\right\rfloor,\ i\not\equiv 0 \pmod n\right\},$$

$$S_4(k)\cup S_7(k)=\left\{\left\{0,\,jgn+i,\,kgn+\frac{gn}{2}\right\}:\ 0\leqslant j\leqslant h-1,\ \left\lfloor\frac{gn}{4}\right\rfloor<i\leqslant\frac{gn}{2},\ i\not\equiv 0 \pmod n\right\},$$
$$S_3(k)\cup S_8(k)=\left\{\left\{0,\,jgn+i,\,kgn+\frac{gn}{2}\right\}:\ 0\leqslant j\leqslant h-1,\ \frac{gn}{2}<i\leqslant\left\lfloor\frac{3gn}{4}\right\rfloor,\ i\not\equiv 0 \pmod n\right\},$$
$$S_2(k)\cup S_5(k)=\left\{\left\{0,\,jgn+i,\,kgn+\frac{gn}{2}\right\}:\ 0\leqslant j\leqslant h-1,\ \left\lfloor\frac{3gn}{4}\right\rfloor<i<gn,\ i\not\equiv 0 \pmod n\right\}.$$

Thus we have

$$\bigcup_{k=0}^{(h-3)/2}\bigcup_{r=1}^{8}S_r(k)=\left\{\left\{0,\,jgn+i,\,kgn+\frac{gn}{2}\right\}:\ 0\leqslant j\leqslant h-1,\ 0\leqslant k\leqslant (h-3)/2,\ 0<i<gn,\ i\not\equiv 0 \pmod n\right\}.$$

Therefore $\mathcal{C}_4'$ and $\mathcal{C}_5'$ cover all triples of form $\{t,t+\xi,t+\eta\}$, where $\xi\not\equiv 0 \pmod n$, $\eta\equiv gn/2 \pmod{gn}$, $\eta\neq hgn/2$ and $0\leqslant t\leqslant hgn-1$ (note that for $\frac{h+1}{2}\leqslant k\leqslant h-1$, it is easy to verify that $S_1(k)=S_3(k-\frac{h+1}{2})$, $S_2(k)=S_4(k-\frac{h+1}{2})$, $S_3(k)=S_1(k-\frac{h+1}{2})$, $S_4(k)=S_2(k-\frac{h+1}{2})$, $S_5(k)=S_7(k-\frac{h+1}{2})$, $S_6(k)=S_8(k-\frac{h+1}{2})$, $S_7(k)=S_5(k-\frac{h+1}{2})$ and $S_8(k)=S_6(k-\frac{h+1}{2})$).

Next we compute the number of blocks in $\mathcal{T}'$. It is easy to see that $|\mathcal{C}_1|=h^2|\mathcal{F}_2|=h^2g(n-1)(gn+g-9)/24$, $|\mathcal{C}_2|=hg(n-1)/4$, $|\mathcal{C}_3|=h(h-1)g(n-1)/4$, $|\mathcal{C}_4|=(h+1)(h-1)g(n-1)/16$ and $|\mathcal{C}_5|=(h-1)^2g(n-1)/16$. Thus

$$|\mathcal{T}'|=hgn\big(|\mathcal{C}_1|+|\mathcal{C}_2|+|\mathcal{C}_3|+|\mathcal{C}_4|+|\mathcal{C}_5|\big)=\frac{h^2gn(gn-g)(hgn+hg-3)}{24}.$$

Since $\mathcal{T}'$ contains the expected number of quadruples, and any triple of form $\{j,j+i,j+2i\}$ or $\{j,j+i,j+hgn/2\}$, $1\leqslant i\leqslant hgn/2$, $i\not\equiv 0 \pmod n$ and $0\leqslant j\leqslant hgn-1$, is contained in a block of $\mathcal{C}_2'$, it suffices to show that each triple not contained in any group appears in at least one quadruple of $\mathcal{T}'$.

Let $T=\{a,b,c\}$ be such a triple. Let $x=b-a$, $y=c-b$ and $z=a-c$, where the arithmetic is modulo $hgn$. Obviously $x+y+z\equiv 0 \pmod{hgn}$ and $x,y,z$ are not all congruent to 0 modulo $n$. Thus it is impossible that $x\equiv y\equiv z \pmod{gn}$. Otherwise, let $x=x_1gn+r$, $y=y_1gn+r$ and $z=z_1gn+r$, $0\leqslant r\leqslant gn-1$. Then we have $(x_1+y_1+z_1)gn+3r\equiv 0 \pmod{hgn}$. It follows that $3r\equiv 0 \pmod{gn}$ and $r=0$, $gn/3$ or $2gn/3$. By Lemma 2.1(2) with $K_T=\{4\}$, if $gn\equiv 0 \pmod 3$, then $g\equiv 0 \pmod 3$. Therefore $gn/3\equiv 0 \pmod n$. Then $x\equiv y\equiv z\equiv 0 \pmod n$. That is a contradiction. In the following all the arithmetics are modulo $hgn$.

*Case* 1. Exactly two of $x$, $y$ and $z$ are equal. Without loss of generality, suppose that $x=y\neq z$. Then $T=\{a,b,c\}=\{a,a+(b-a),a+2(b-a)\}$. It is easy to see that $\mathcal{C}_2'$ contains all triples of form $\{j,j+i,j+2i\}$ and $\{j,j+i,j+hgn/2\}$, where $1\leqslant i\leqslant hgn$, $i\not\equiv 0 \pmod n$ and $0\leqslant j\leqslant hgn-1$. Since $b-a\not\equiv 0 \pmod n$, $T$ is contained in $\mathcal{C}_2'$.

*Case* 2. $x$, $y$ and $z$ are pairwise distinct.

(1) Suppose that one of $x$, $y$ and $z$ is divisible by $gn/2$. Without loss of generality, let $z\equiv 0 \pmod{gn/2}$.

If $z=jgn$, $1\leqslant j\leqslant h-1$, it is easy to verify that if $x<y$, $x<(h-j)gn/2$ and if $y<x$, $y<(h-j)gn/2$. Then $T$ is contained in $\{c+a-b,c+a-b+x,c+a-b+x+z,c+a-b+2x+z\}=\{c+a-b,c,a,b\}$ from $\mathcal{C}_3'$ if $x<y$, or in $\{b,b+y,b+y+z,b+2y+z\}=\{b,c,a,c-b+a\}$ from $\mathcal{C}_3'$ if $y<x$.

If $z=hgn/2$, then $T=\{a,b,c\}=\{c,c+(b-c),c+hgn/2\}$. Note that due to $z\equiv 0 \pmod n$, $b-c\not\equiv 0 \pmod n$. Since $\mathcal{C}_2'$ contains all triples of form $\{j,j+i,j+hgn/2\}$, where $1\leqslant i\leqslant hgn$, $i\not\equiv 0 \pmod n$ and $0\leqslant j\leqslant hgn-1$, $T$ is contained in $\mathcal{C}_2'$.

If $z\equiv gn/2 \pmod{gn}$ and $z\neq hgn/2$, there exists a block $B\in\mathcal{C}_4'\cup\mathcal{C}_5'$ covering $\{a,a+x,a-z\}=\{a,b,c\}$. Thus $T$ is contained in a block from $\mathcal{C}_4'\cup\mathcal{C}_5'$.

(2) None of $x$, $y$ and $z$ is divisible by $gn/2$.

If exactly two of $x$, $y$ and $z$ are equal modulo $gn$, without loss of generality, let $y\equiv z\not\equiv x \pmod{gn}$. Let $y=y_1gn+r$ and $z=z_1gn+r$, $0\leqslant y_1,z_1\leqslant h-1$, $1\leqslant r\leqslant gn-1$, $r\neq gn/2$. Note that $y_1<z_1$

if $y<z$ and $z_1<y_1$ if $z<y$. Then if $y+z<hgn$, $T$ is contained in $\{b, b+y_1gn+r, b+(z_1-y_1)gn+y_1gn+r, b+(z_1-y_1)gn+2(y_1gn+r)\}=\{b, b+y, b+z, b+z+y\}=\{b, c, a-c+b, a\}$ from $\mathcal{C}_3'$ when $y<z$, or in $\{b, b+z_1gn+r, b+(y_1-z_1)gn+z_1gn+r, b+(y_1-z_1)gn+2(z_1gn+r)\}=\{b, b+z, b+y, b+y+z\}=\{b, b+a-c, c, a\}$ from $\mathcal{C}_3'$ when $z<y$. If $y+z>hgn$, $T$ is contained in $\{a, a+(h-z_1)gn-r, a+(z_1-y_1)gn+(h-z_1)gn-r, a+(z_1-y_1)gn+2((h-z_1)gn-r)\}=\{a, a-z, a-y, a-y-z\}=\{a, c, a-c+b, b\}$ from $\mathcal{C}_3'$ when $y<z$, or in $\{a, a+(h-y_1)gn-r, a+(y_1-z_1)gn+(h-y_1)gn-r, a+(y_1-z_1)gn+2((h-y_1)gn-r)\}=\{a, a-c+b, c, b\}$ from $\mathcal{C}_3'$ when $z<y$.

Otherwise, $x\not\equiv y\not\equiv z\not\equiv x \pmod{gn}$. Write $s=e_sgn+f_s$, where $0\leqslant e_s\leqslant h-1$, $1\leqslant f_s\leqslant gn-1$ and $s\in\{a,b,c\}$. In the given cyclic 0-FG*$(3,(\emptyset,4),gn)$ of type $g^n$, there exists a base block $B=\{x_a,x_b,x_c,x_d\}$ and an element $j$, $0\leqslant j\leqslant gn-1$, such that $x_s+j=f_s+\sigma_sgn$, where $\sigma_s\in\{0,1\}$ and $s\in\{a,b,c\}$. Then there exists a base block $\{(x_a,y_a),(x_b,y_b),(x_c,y_c),(x_d,y_d)\}$ in the set $\mathcal{A}_B$ of base blocks of a semi-cyclic $H(4,h,4,3)$ and an element $j'$, $0\leqslant j'\leqslant h-1$, such that $y_s+j'=e_s-\sigma_s+\sigma_s'h$, $s\in\{a,b,c\}$ and $\sigma_s'\in\{0,1\}$. This yields the fact that there exists a block $B'=\{x_a+y_agn, x_b+y_bgn, x_c+y_cgn, x_d+y_dgn\}\in\mathcal{C}_1$ such that $B'+(j'gn+j)$ contains $\{a,b,c\}$.

The proof is completed. □

**Lemma 4.2.** *There exists a strictly cyclic 0-FG*$^*(3,(\emptyset,4),gn)$ *of type* $g^n$ *for each* $(g,n)\in\{(8,2),\ (16,2),\ (4,5),\ (8,5)\}$.

**Proof.** The required designs are constructed on $Z_{gn}$ with groups $\{\{in+j\colon\ 0\leqslant i\leqslant g-1\}\colon\ 0\leqslant j\leqslant n-1\}$. All base blocks are divided into two parts. The first part contains base blocks $\{0, i, 2i, \frac{gn}{2}+i\}$, $1\leqslant i\leqslant gn/4$ and $i\not\equiv 0 \pmod n$. The second part is generated by the following base blocks. All other blocks are obtained by developing these base blocks by $+1$ modulo $gn$.

$(g,n)=(8,2)$:
$\{0,1,3,4\}$, $\{0,1,5,6\}$, $\{0,2,5,7\}$, $\{0,4,7,11\}$, $\{0,6,7,13\}$.

$(g,n)=(16,2)$:
$\{0,1,7,8\}$, $\{0,1,9,10\}$, $\{0,1,11,12\}$, $\{0,1,13,14\}$, $\{0,1,15,18\}$,
$\{0,1,27,28\}$, $\{0,1,29,30\}$, $\{0,2,5,7\}$, $\{0,2,9,11\}$, $\{0,2,13,15\}$,
$\{0,3,7,10\}$, $\{0,3,8,11\}$, $\{0,3,9,14\}$, $\{0,3,12,15\}$, $\{0,3,13,22\}$,
$\{0,3,21,26\}$, $\{0,4,9,21\}$, $\{0,4,11,25\}$, $\{0,4,13,17\}$, $\{0,4,15,27\}$,
$\{0,5,12,25\}$, $\{0,5,13,18\}$, $\{0,5,15,22\}$, $\{0,6,13,21\}$, $\{0,6,15,23\}$,
$\{0,6,17,25\}$.

$(g,n)=(4,5)$:
$\{0,1,3,4\}$, $\{0,1,5,16\}$, $\{0,1,6,7\}$, $\{0,1,8,13\}$, $\{0,1,9,12\}$,
$\{0,2,5,17\}$, $\{0,2,6,8\}$, $\{0,2,7,9\}$, $\{0,3,7,16\}$, $\{0,3,9,14\}$.

$(g,n)=(8,5)$:
$\{0,1,3,4\}$, $\{0,1,5,6\}$, $\{0,1,7,8\}$, $\{0,1,9,10\}$, $\{0,1,11,12\}$,
$\{0,1,13,14\}$, $\{0,1,15,16\}$, $\{0,1,17,18\}$, $\{0,1,19,22\}$, $\{0,2,5,7\}$,
$\{0,2,6,8\}$, $\{0,2,9,11\}$, $\{0,2,10,12\}$, $\{0,2,13,15\}$, $\{0,2,14,16\}$,
$\{0,2,17,19\}$, $\{0,2,18,24\}$, $\{0,3,7,10\}$, $\{0,3,8,11\}$, $\{0,3,9,12\}$,
$\{0,3,13,16\}$, $\{0,3,14,17\}$, $\{0,3,15,18\}$, $\{0,3,19,24\}$, $\{0,4,9,13\}$,
$\{0,4,10,14\}$, $\{0,4,11,15\}$, $\{0,4,12,16\}$, $\{0,4,17,21\}$, $\{0,4,18,25\}$,
$\{0,4,19,26\}$, $\{0,5,11,19\}$, $\{0,5,12,17\}$, $\{0,5,13,32\}$, $\{0,5,14,31\}$,
$\{0,5,16,29\}$, $\{0,5,18,23\}$, $\{0,5,26,34\}$, $\{0,6,13,25\}$, $\{0,6,15,31\}$,
$\{0,6,16,30\}$, $\{0,6,17,29\}$, $\{0,6,18,27\}$, $\{0,6,19,28\}$, $\{0,6,21,33\}$,
$\{0,7,15,29\}$, $\{0,7,16,23\}$, $\{0,7,17,30\}$, $\{0,7,18,32\}$, $\{0,8,17,25\}$,
$\{0,8,18,30\}$, $\{0,9,19,30\}$. □

**Lemma 4.3.** *There exists a strictly cyclic 0-FG*$^*(3,(\emptyset,4),64)$ *of type* $32^2$.

**Proof.** A strictly cyclic 0-FG$^*(3, (\emptyset, 4), 64)$ of type $32^2$ is constructed on $Z_{64}$ with groups $\{\{2k+j\colon\ 0 \leqslant k \leqslant 31\}\colon\ 0 \leqslant j \leqslant 1\}$. All base blocks are divided into three parts. The first part contains base blocks $\{0, i, 2i, 32+i\}$, $1 \leqslant i \leqslant 16$ and $i \not\equiv 0 \pmod 2$.

Let $\mathcal{B} = \{\{b_{k1}, b_{k2}, b_{k3}, b_{k4}\}\colon\ 1 \leqslant k \leqslant 5\}$ be the family of base blocks of a strictly cyclic 0-FG$^*(3, (\emptyset, 4), 16)$ of type $8^2$, which is given by Lemma 4.2. Let $M = (a_{ij})$ be a $2 \times 4$ 3-simple matrix over $Z_2$ as follows (see [3]):

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then the second part consists of the following 80 base blocks:

$$\big\{\{16b_{k1} + a_{1i} + 2a_{1j}, 16b_{k2} + a_{2i} + 2a_{2j}, 16b_{k3} + a_{3i} + 2a_{3j}, 16b_{k4} + a_{4i} + 2a_{4j}\},\ 1 \leqslant i, j \leqslant 4,\ 1 \leqslant k \leqslant 5\big\}.$$

All the remaining 36 base blocks are listed as follows:

| | | | | |
|---|---|---|---|---|
| {0, 1, 8, 9}, | {0, 1, 15, 16}, | {0, 1, 17, 18}, | {0, 1, 24, 31}, | {0, 1, 25, 40}, |
| {0, 1, 34, 41}, | {0, 2, 9, 49}, | {0, 2, 17, 57}, | {0, 2, 25, 41}, | {0, 3, 8, 11}, |
| {0, 3, 13, 16}, | {0, 3, 19, 22}, | {0, 3, 24, 29}, | {0, 3, 27, 40}, | {0, 3, 38, 43}, |
| {0, 5, 11, 16}, | {0, 5, 13, 42}, | {0, 5, 21, 48}, | {0, 5, 24, 45}, | {0, 5, 27, 56}, |
| {0, 6, 19, 27}, | {0, 6, 43, 51}, | {0, 7, 15, 46}, | {0, 7, 16, 23}, | {0, 7, 25, 56}, |
| {0, 8, 17, 25}, | {0, 8, 19, 29}, | {0, 8, 23, 31}, | {0, 8, 43, 53}, | {0, 9, 23, 40}, |
| {0, 9, 25, 34}, | {0, 9, 33, 50}, | {0, 11, 24, 35}, | {0, 11, 27, 38}, | {0, 13, 29, 48}, |
| {0, 15, 31, 48}. □ | | | | |

**Lemma 4.4.** *There exists a strictly cyclic* 0-*FG*$^*(3, (\emptyset, 4), 3g)$ *of type* $g^3$ *for* $g \in \{12, 24\}$.

**Proof.** The required designs are constructed on $Z_{3g}$ with groups $\{\{3i+j\colon\ 0 \leqslant i \leqslant g-1\}\colon\ 0 \leqslant j \leqslant 2\}$. All base blocks for these designs can be divided into two parts. The first part contains base blocks $\{0, i, 2i, \frac{gn}{2}+i\}$, $1 \leqslant i \leqslant gn/4$ and $i \not\equiv 0 \pmod 3$. The second part is generated by multiplying each of the following base blocks by $25^i$, $i = 0, 1, 2$. All other blocks are obtained by developing these base blocks by $+1$ modulo $3g$.

| | | | | | |
|---|---|---|---|---|---|
| $g = 12$: | {0, 1, 3, 4}, | {0, 1, 5, 8}, | {0, 1, 6, 7}, | {0, 1, 9, 10}, | {0, 1, 11, 12}, |
| | {0, 1, 15, 16}, | {0, 1, 17, 29}, | {0, 1, 20, 32}, | {0, 2, 5, 7}, | {0, 2, 6, 8}, |
| | {0, 2, 9, 17}, | {0, 2, 10, 12}, | {0, 2, 21, 29}. | | |

| | | | | | |
|---|---|---|---|---|---|
| $g = 24$: | {0, 1, 3, 4}, | {0, 1, 5, 6}, | {0, 1, 7, 8}, | {0, 1, 9, 10}, | {0, 1, 11, 12}, |
| | {0, 1, 13, 14}, | {0, 1, 15, 16}, | {0, 1, 17, 18}, | {0, 1, 19, 20}, | {0, 1, 21, 22}, |
| | {0, 1, 23, 24}, | {0, 1, 27, 28}, | {0, 1, 29, 30}, | {0, 1, 31, 32}, | {0, 1, 33, 34}, |
| | {0, 1, 35, 38}, | {0, 2, 5, 7}, | {0, 2, 6, 8}, | {0, 2, 9, 11}, | {0, 2, 10, 12}, |
| | {0, 2, 13, 15}, | {0, 2, 14, 16}, | {0, 2, 17, 19}, | {0, 2, 18, 20}, | {0, 2, 21, 41}, |
| | {0, 2, 22, 24}, | {0, 2, 29, 31}, | {0, 2, 30, 32}, | {0, 2, 33, 53}, | {0, 2, 34, 39}, |
| | {0, 2, 35, 40}, | {0, 3, 7, 40}, | {0, 3, 8, 17}, | {0, 3, 10, 19}, | {0, 3, 11, 44}, |
| | {0, 4, 9, 13}, | {0, 4, 10, 18}, | {0, 4, 11, 31}, | {0, 4, 12, 16}, | {0, 4, 14, 62}, |
| | {0, 4, 15, 19}, | {0, 4, 20, 24}, | {0, 4, 34, 42}, | {0, 4, 35, 59}, | {0, 4, 41, 65}, |
| | {0, 4, 58, 66}, | {0, 5, 11, 18}, | {0, 5, 12, 43}, | {0, 5, 13, 53}, | {0, 5, 15, 32}, |
| | {0, 5, 16, 48}, | {0, 5, 19, 60}, | {0, 5, 21, 56}, | {0, 5, 35, 42}, | {0, 5, 45, 62}, |
| | {0, 5, 59, 66}, | {0, 7, 21, 58}, | {0, 8, 35, 45}. □ | | |

**Lemma 4.5.** *There exists a strictly cyclic* 0-*FG*$^*(3, (\emptyset, 4), 144)$ *of type* $48^3$.

**Proof.** The required design is constructed on $Z_{144}$ with groups $\{\{3i+j\colon\ 0 \leqslant i \leqslant 47\}\colon\ 0 \leqslant j \leqslant 2\}$. All base blocks are divided three parts. The first part consists of all base blocks of a strictly cyclic 0-FG*$(3, (\emptyset, 4), 72)$ of type $24^3$ on $\{0, 2, 4, \ldots, 142\}$. The second part contains base blocks $\{0, i, 2i, \frac{gn}{2}+i\}$, $1 \leqslant i \leqslant 36$, $i \equiv 1, 5 \pmod 6$. The third part is generated by multiplying each of the following base blocks by $7^i$, $i = 0, 1, 2, 3, 4, 5$, where the base blocks with a star will generate three distinct blocks. All other blocks are obtained by developing these base blocks by $+1$ modulo 144.

| | | | | |
|---|---|---|---|---|
| $\{0,1,55,56\}^*$, | $\{0,5,13,18\}^*$, | $\{0,11,29,40\}^*$, | $\{0,17,71,88\}^*$, | $\{0,10,69,85\}$, |
| $\{0,1,3,4\}$, | $\{0,1,5,6\}$, | $\{0,1,7,8\}$, | $\{0,1,9,10\}$, | $\{0,1,11,12\}$, |
| $\{0,1,13,14\}$, | $\{0,1,15,16\}$, | $\{0,1,17,18\}$, | $\{0,1,19,20\}$, | $\{0,1,21,22\}$, |
| $\{0,1,23,24\}$, | $\{0,1,25,26\}$, | $\{0,1,27,28\}$, | $\{0,1,29,30\}$, | $\{0,1,31,32\}$, |
| $\{0,1,33,34\}$, | $\{0,1,35,36\}$, | $\{0,1,37,38\}$, | $\{0,1,39,40\}$, | $\{0,1,43,44\}$, |
| $\{0,1,45,46\}$, | $\{0,1,47,48\}$, | $\{0,1,51,52\}$, | $\{0,1,53,54\}$, | $\{0,1,57,58\}$, |
| $\{0,1,59,60\}$, | $\{0,1,61,62\}$, | $\{0,1,63,64\}$, | $\{0,1,65,66\}$, | $\{0,1,67,68\}$, |
| $\{0,1,69,70\}$, | $\{0,1,71,74\}$, | $\{0,2,5,15\}$, | $\{0,5,64,69\}$, | $\{0,2,11,13\}$, |
| $\{0,2,17,19\}$, | $\{0,2,21,23\}$, | $\{0,2,25,27\}$, | $\{0,2,29,31\}$, | $\{0,2,33,35\}$, |
| $\{0,2,37,39\}$, | $\{0,2,45,53\}$, | $\{0,91,99,142\}$, | $\{0,2,59,61\}$, | $\{0,2,63,65\}$, |
| $\{0,2,67,69\}$, | $\{0,2,71,75\}$, | $\{0,3,8,11\}$, | $\{0,3,16,19\}$, | $\{0,3,20,29\}$, |
| $\{0,115,124,141\}$, | $\{0,3,23,124\}$, | $\{0,4,9,13\}$, | $\{0,4,15,19\}$, | $\{0,4,23,27\}$, |
| $\{0,4,29,33\}$, | $\{0,4,31,57\}$, | $\{0,87,113,140\}$, | $\{0,4,35,67\}$, | $\{0,4,61,87\}$, |
| $\{0,108,133,139\}$, | $\{0,4,39,77\}$, | $\{0,67,105,140\}$, | $\{0,4,43,69\}$, | $\{0,8,19,27\}$, |
| $\{0,77,109,140\}$, | $\{0,4,65,83\}$, | $\{0,5,11,36\}$, | $\{0,4,37,111\}$, | $\{0,5,17,40\}$, |
| $\{0,104,127,139\}$, | $\{0,5,22,53\}$, | $\{0,91,122,139\}$, | $\{0,5,24,29\}$, | $\{0,5,30,35\}$, |
| $\{0,5,32,117\}$, | $\{0,5,37,42\}$, | $\{0,5,43,106\}$, | $\{0,5,59,78\}$, | $\{0,66,85,139\}$, |
| $\{0,129,139,142\}$, | $\{0,6,17,29\}$, | $\{0,115,127,138\}$, | $\{0,8,17,75\}$, | $\{0,69,127,136\}$, |
| $\{0,75,101,140\}$, | $\{0,8,23,87\}$, | $\{0,57,121,136\}$, | $\{0,10,27,37\}$, | $\{0,10,29,125\}$. □ |

**Lemma 4.6.** *There exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), 2^n)$ *of type* $(2^{n-1})^2$ *for any integer* $n \geqslant 4$.

**Proof.** By Lemmas 4.2 and 4.3, we have the cases of $n = 4$, 5, 6. For $n \geqslant 7$, start with a strictly cyclic 0-FG$(3, (\emptyset, 4), 16)$ of type $8^2$. Applying Construction 2.4 inductively we have a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{3a+4})$ of type $(8 \cdot 8^a)^2$ for all $a \geqslant 0$. Take a strictly cyclic 0-FG$(3, (\emptyset, 4), 32)$ of type $16^2$. Then applying Construction 2.4 inductively again with the resulting strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{3a+4})$ of type $(8 \cdot 8^a)^2$, we have a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{3a+4b+4})$ of type $(8 \cdot 8^a 16^b)^2$ for all $a, b \geqslant 0$. It is easily checked that for any $n \geqslant 7$ and $n \neq 9$ there exist non-negative integers $a$, $b$ such that $n = 3a + 4b + 4$. It follows that there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^n)$ of type $(2^{n-1})^2$ for $n \neq 9$. To deal with the case of $n = 9$, start with a strictly cyclic 0-FG$(3, (\emptyset, 4), 64)$ of type $32^2$. Apply Construction 2.4 with a strictly cyclic 0-FG$(3, (\emptyset, 4), 16)$ of type $8^2$ to complete the proof. □

**Lemma 4.7.** *There exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), 2g)$ *of type* $g^2$ *for any integer* $g \equiv 0 \pmod 8$.

**Proof.** Assume that $g = 8 \cdot 2^n h$, where $n \geqslant 0$ is an integer and $h \geqslant 1$ is an odd integer. Start with a strictly cyclic 0-FG*$(3, (\emptyset, 4), 16)$ of type $8^2$ from Lemma 4.2. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 16h)$ of type $(8h)^2$ for any odd positive integer $h$. By Lemma 4.6 there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{n+1})$ of type $(2^n)^2$ for any integer $n \geqslant 3$. Then apply Construction 2.4 to obtain a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{n+4}h)$ of type $(8 \cdot 2^n h)^2$. Thus we have the cases of $n = 0$ and $n \geqslant 3$.

For the cases of $n = 1$, 2, there exist a strictly cyclic 0-FG*$(3, (\emptyset, 4), 32)$ of type $16^2$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 64)$ of type $32^2$ by Lemma 4.2. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 32h)$ of type $(16h)^2$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 64h)$ of type $(32h)^2$ for any odd positive integer $h$. This completes the proof. □

**Lemma 4.8.** *There exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), 3g)$ *of type* $g^3$ *for any integer* $g \equiv 0 \pmod{12}$.

**Proof.** Assume that $g = 12 \cdot 2^n h$, where $n \geqslant 0$ is an integer and $h \geqslant 1$ is an odd integer. Start with a strictly cyclic 0-FG*$(3, (\emptyset, 4), 36)$ of type $12^3$ from Lemma 4.4. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 36h)$ of type $(12h)^3$ for any odd positive integer $h$. By Lemma 4.6 there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{n+1})$ of type $(2^n)^2$ for any integer $n \geqslant 3$. Then apply Construction 2.4 to obtain a strictly cyclic 0-FG$(3, (\emptyset, 4), 36 \cdot 2^n h)$ of type $(12 \cdot 2^n h)^3$. Thus we have the cases of $n = 0$ and $n \geqslant 3$.

For the cases of $n = 1$, 2, there exist a strictly cyclic 0-FG*$(3, (\emptyset, 4), 72)$ of type $24^3$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 144)$ of type $48^3$ by Lemmas 4.4 and 4.5, respectively. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 72h)$ of type $(24h)^3$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 144h)$ of type $(48h)^3$ for any odd positive integer $h$. This completes the proof. □

**Lemma 4.9.** *There exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), 5g)$ *of type* $g^5$ *for any integer* $g \equiv 0 \pmod 2$.

**Proof.** Assume that $g = 2 \cdot 2^n h$, where $n \geqslant 0$ is an integer and $h \geqslant 1$ is an odd integer. Start with a strictly cyclic 0-FG*$(3, (\emptyset, 4), 10)$ of type $2^5$, which is equivalent to an sSQS(10). The unique sSQS(10) up to isomorphism can be found in Lemma 5.3. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 10h)$ of type $(2h)^5$ for any odd positive integer $h$. By Lemma 4.6 there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{n+1})$ of type $(2^n)^2$ for any integer $n \geqslant 3$. Then apply Construction 2.4 to obtain a strictly cyclic 0-FG$(3, (\emptyset, 4), 5 \cdot 2^{n+1} h)$ of type $(2 \cdot 2^n h)^5$. Thus we have the cases of $n = 0$ and $n \geqslant 3$.

For the cases of $n = 1$ and 2, there exist a strictly cyclic 0-FG*$(3, (\emptyset, 4), 20)$ of type $4^5$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 40)$ of type $8^5$ by Lemma 4.2. Applying Construction 4.1 we have a strictly cyclic 0-FG*$(3, (\emptyset, 4), 20h)$ of type $(4h)^5$ and a strictly cyclic 0-FG*$(3, (\emptyset, 4), 40h)$ of type $(8h)^5$ for any odd positive integer $h$. This completes the proof. □

Note that actually Lemmas 4.7, 4.8 and 4.9 give the necessary and sufficient conditions for the existence of a strictly cyclic 0-FG$(3, (\emptyset, 4), gn)$ of type $g^n$ for $n = 2$, 3, 5.

**Lemma 4.10.** *Suppose that there exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), gn)$ *of type* $g^n$ *and* $g \neq 1$. *Then there exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), hgn)$ *of type* $(hg)^n$ *for any positive integer* $h \equiv 0 \pmod 8$.

**Proof.** There exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2h)$ of type $h^2$ and a semi-cyclic $H(4, h, 4, 3)$ for any positive integer $h \equiv 0 \pmod 8$ by Lemma 4.7 and Corollary 3.5, respectively. By Lemma 2.1(3), $g$ is even. Then apply Construction 2.4 to complete the proof. □

**Lemma 4.11.** *Let* $g \equiv 0 \pmod 8$. *Suppose that there exists a strictly cyclic* $H(n, g, K, 3)$. *Then there exists a strictly cyclic* 0-*FG*$(3, (\emptyset, K \cup \{4\}), gn)$ *of type* $h^m$, *where* $(h, m) = (g, n)$ *when* $n \equiv 1 \pmod 2$ *and* $(h, m) = (2g, n/2)$ *when* $n \equiv 0 \pmod 2$.

**Proof.** By Lemma 4.7, there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), 2g)$ of type $g^2$ for any positive integer $g \equiv 0 \pmod 8$. Then apply Construction 2.5 to complete the proof. □

Combining the results of Lemmas 3.12 and 4.11, we have

**Corollary 4.12.** *If there exists a* CSQS$(n)$ *for* $n \equiv 2, 10 \pmod{12}$, *then for any* $g \equiv 0 \pmod 8$, *there exists a strictly cyclic* 0-*FG*$(3, (\emptyset, 4), gn)$ *of type* $(2g)^{n/2}$.

Strictly cyclic 0-FG$(3, (\emptyset, 4), gn)$'s of type $g^n$ can be also obtained from a special kind of SQS$(v)$ called a rotational SQS$(v)$. A *rotational* SQS$(v)$ is an SQS$(v)$ with an automorphism consisting of one fixed point and a cycle of length $v - 1$. Such a system is denoted by RoSQS$(v)$. It is known that the necessary condition for the existence of an RoSQS$(v+1)$ is $v \equiv 1, 3 \pmod 6$ and $v \neq 9, 13$ [17]. It is easily verified that if there exists an RoSQS$(v+1)$ for $v \equiv 1 \pmod 6$, then so does a strictly cyclic 1-fan S$(3, (3, 4), v)$.

**Theorem 4.13.** *(See [17].)*

(1) *There exists an* RoSQS$(v+1)$ *for all* $v \equiv 1, 3 \pmod 6$ *with* $4 \leqslant v < 100$ *except for* $v \in \{9, 13\}$ *and possibly for* $v \in \{45, 55, 69, 81, 85, 91, 97\}$.
(2) *There exists an* RoSQS$(uv+1)$ *for any* $u \in \{4^n - 1\colon$ *integer* $n \geqslant 1\} \cup \{1, 27, 33, 39, 51, 87\}$ *and* $v$ *is an integer taken from set* $P = \{p \equiv 7 \pmod{12}\colon$ *p is a prime*$\} \cup \{2^n - 1\colon$ *odd integer* $n > 1\} \cup \{25, 37, 61, 73\}$, *or a product of such integers.*

**Lemma 4.14.** *Let* $g \equiv 0 \pmod{12}$. *Suppose that there exists an* RoSQS$(v+1)$. *Then there exists a strictly cyclic* 0-FG$(3, (\emptyset, 4), hm)$ *of type* $h^m$, *where* $(h, m) = (g, v)$ *when* $v \equiv 1 \pmod 6$ *and* $(h, m) = (3g, v/3)$ *when* $v \equiv 3 \pmod 6$.

**Proof.** When $v \equiv 1 \pmod 6$, since the existence of an RoSQS$(v+1)$ implies the existence of a strictly cyclic 1-fan $S(3, (3, 4), v)$, by Construction 2.3 the required strictly cyclic 0-FG$(3, (\emptyset, 4), gv)$ of type $g^v$ is obtained, where the needed strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 3g)$ of type $g^3$ is from Lemma 4.8 and semi-cyclic $H(4, g, 4, 3)$ is from Corollary 3.5.

When $v \equiv 3 \pmod 6$, suppose that the given RoSQS$(v+1)$ is based on $Z_v \cup \{\infty\}$ with base block set $\mathcal{B} = \mathcal{B}_\infty \cup \mathcal{B}_0$, where $\mathcal{B}_\infty$ (or $\mathcal{B}_0$) consists of all base blocks (or not) containing $\infty$. Note that $\mathcal{B}_\infty$ contains a special base block $\{\infty, 0, v/3, 2v/3\}$. Let $\mathcal{G} = \{\{j, v/3 + j, 2v/3 + j\}\colon\ 0 \leqslant j < v/3\}$. Let $\mathcal{B}' = \{B - \{\infty\}\colon\ B \in \mathcal{B}_\infty\} \setminus \{\{0, v/3, 2v/3\}\}$. Then it is readily checked that $(Z_v, \mathcal{G}, \mathcal{B}', \mathcal{B}_0)$ is a strictly cyclic 1-FG$(3, (3, 4), v)$ of type $3^{v/3}$. Applying Construction 2.3, we have a strictly cyclic 0-FG$(3, (\emptyset, 4), gv)$ of type $(3g)^{v/3}$, where the needed strictly semi-cyclic 0-FG$(3, (\emptyset, 4), 3g)$ of type $g^3$ is from Lemma 4.8 and semi-cyclic $H(4, g, 4, 3)$ is from Corollary 3.5. □

## 5. Constructions for sSQS's

In this section we give a recursive construction for sSQS's. Making use of this construction, some infinite families of sSQS's are obtained.

In an sSQS$(v)$, if any triple of form $\{j, j+i, j+2i\}$ or $\{j, j+i, j+\frac{v}{2}\}$, where $1 \leqslant i < v/2$ and $0 \leqslant j \leqslant v-1$, is contained in the block $\{j, j+a, j+2a, j+\frac{v}{2}+a\}$ for some $1 \leqslant a \leqslant (v-2)/4$, then such an sSQS$(v)$ is denoted by sSQS$^*(v)$.

**Construction 5.1.** *Suppose that there exist an* sSQS$^*(2u)$ *and an* sSQS$^*(2v)$. *Then there exists an* sSQS$^*(2uv)$.

**Proof.** It is easy to see that the existence of an sSQS$^*(2u)$ is equivalent to a cyclic 0-FG$^*(3, (\emptyset, 4), 2u)$ of type $2^u$. Since $v \equiv 1$ or $5 \pmod{12}$ is an odd integer, by Construction 4.1 we have a cyclic 0-FG$^*(3, (\emptyset, 4), 2uv)$ of type $(2v)^u$. Apply Construction 2.2 with the given sSQS$^*(2v)$ to complete the proof. □

**Corollary 5.2.** *Let n be a positive integer. If there exists an* sSQS$^*(2p)$ *for* $p \equiv 1$ *or* $5 \pmod{12}$. *Then there exists an* sSQS$^*(2p^n)$.

Corollary 5.2 is analogous to Theorem 3.3 in [26], which deals with the cases of $p \equiv 5 \pmod{12}$.

In [20] Köhler gave an investigation on the existence of an sSQS$^*(v)$, especially for $v = 2p$, $p$ a prime and $p \equiv 5 \pmod{12}$. He introduced a special kind of graph and proved that if this special kind of graph has a 1-factor, then there exists an sSQS$^*(v)$. Siemon continued the study in [25–28] based on the methods from Köhler. For more details on this method, the reader may refer to [1].

**Lemma 5.3.** *(See [20,25,27].)*

(1) *There exists an* sSQS$^*(v)$ *for* $v \in \{2, 10, 26, 34, 50, 58, 74, 82, 106, 178, 202, 226, 274, 298, 346, 394, 466, 586, 634\}$.
(2) *There exists an* sSQS$^*(v)$ *for* $v \in \{122, 170, 194, 314, 338, 386, 458, 578\}$.
(3) *There exists an* sSQS$^*(2p)$ *for any prime* $p \equiv 53, 77 \pmod{120}$ *and* $p < 500000$.

In [26] Siemon pointed out that Piotrowski [24] proved the following theorem:

**Lemma 5.4.** *(See [26].)*

(1) *There exists an* sSQS$(2p)$ *for any prime* $p \equiv 1 \pmod 4$ *and* $p \leqslant 229$.
(2) *There exists an* sSQS$(2p)$ *for any prime* $p \equiv 1 \pmod 4$, $p \not\equiv 1, 49 \pmod{60}$ *and* $p < 15000$.

In Example 7.7, we give a direct construction for the existence of an sSQS$^*(98)$. Combining Construction 5.1, Lemma 5.3 and Example 7.7, we have the following theorem.

**Theorem 5.5.** *Let* $S_1$ *be the set of all primes* $p \equiv 53, 77 \pmod{120}$ *and* $p < 500000$. *Let* $S_2 = \{5, 13, 17, 25, 29, 37, 41, 49, 61, 85, 89, 97, 101, 113, 137, 149, 157, 169, 193, 229, 233, 289\}$. *Let* $S = S_1 \cup S_2$ *and* $u$ *be any product of integers from the set* $S$. *Then there exists an* sSQS$^*(2u)$.

Summarizing the results of Lemma 5.4 and Theorem 5.5, we have

**Theorem 5.6.** *There exists an* sSQS$(v)$ *for all* $v \equiv 2, 10 \pmod{24}$ *with* $2 \leqslant v \leqslant 650$ *with the possible exceptions of* $v \in \{154, 242, 266, 322, 418, 434, 482, 506, 602\}$.

## 6. Applications to optimal optical orthogonal codes

An optical orthogonal code is a family of sequences with good auto- and cross-correlation properties. Its study has been motivated by an application in a fiber-optical code-division multiple access (CDMA) channel (cf. [6]).

Let $v$, $k$ and $\lambda$ be positive integers. A $(v, k, \lambda)$ *optical orthogonal code* (briefly $(v, k, \lambda)$-OOC), $\mathcal{C}$, is a family of $(0, 1)$ sequences (called *codewords*) of length $v$ and weight $k$ satisfying the following two properties (all subscripts are reduced modulo $v$):

(1) *The auto-correlation property*: for any $\mathbf{x} = \{x_i\}_{i=0}^{v-1} \in \mathcal{C}$ and any integer $r$, $r \not\equiv 0 \pmod v$,

$$\sum_{i=0}^{v-1} x_i x_{i+r} \leqslant \lambda.$$

(2) *The cross-correlation property*: for any $\mathbf{x} = \{x_i\}_{i=0}^{v-1} \in \mathcal{C}$, $\mathbf{y} = \{y_i\}_{i=0}^{v-1} \in \mathcal{C}$ with $\mathbf{x} \neq \mathbf{y}$ and any integer $r$,

$$\sum_{i=0}^{v-1} x_i y_{i+r} \leqslant \lambda.$$

Analogous to the Johnson bound [18] for constant-weight error-correcting codes, the size of a $(v, k, \lambda)$-OOC is upper-bounded (cf. [8]) by

$$\left\lfloor \frac{1}{k} \left\lfloor \frac{v-1}{k-1} \left\lfloor \frac{v-2}{k-2} \left\lfloor \cdots \left\lfloor \frac{v-\lambda}{k-\lambda} \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor \right\rfloor. \tag{6.1}$$

A $(v, k, \lambda)$-OOC with this number of codewords is said to be *optimal*.

OOCs are very closely related to some combinatorial configurations called strictly cyclic packings. A $t$-$(v, k, 1)$ *packing design* is a pair $(X, \mathcal{B})$, where $X$ is a set of $v$ points and $\mathcal{B}$ is a set of subsets of $X$ (called *blocks*), each of cardinality $k$, such that every $t$-subset of $X$ occurs in at most one block.

An *automorphism group* of a packing design $(X, \mathcal{B})$ is a permutation group on $X$ leaving $\mathcal{B}$ invariant. A $t$-$(v, k, 1)$ packing design is said to be *cyclic* if it admits an automorphism consisting of a cycle of length $v$. If the stabilizer of any block of a cyclic $t$-$(v, k, 1)$ packing design is trivial, then the packing design is called *strictly cyclic*. A strictly cyclic $t$-$(v,k,1)$ packing design is called *optimal* if it contains

$$\left\lfloor \frac{1}{k}\left\lfloor \frac{v-1}{k-1}\left\lfloor \frac{v-2}{k-2}\left\lfloor \cdots \left\lfloor \frac{v-t+1}{k-t+1}\right\rfloor \cdots \right\rfloor\right\rfloor\right\rfloor\right\rfloor \tag{6.2}$$

base blocks (cf. [8]).

**Lemma 6.1.** *(See [8].) An optimal $(v,k,\lambda)$-OOC is equivalent to an optimal strictly cyclic $(\lambda+1)$-$(v,k,1)$-packing design, provided that $\lambda < k$ holds.*

In this article we only consider the case of $k=4$ and $\lambda=2$. Combining results from [3] and [4] the following theorem on optimal $(v,4,2)$-OOC is given.

**Theorem 6.2.** *(See [3,4].)*

(1) *There exists an optimal $(v,4,2)$-OOC for all $7 \leqslant v \leqslant 44$ with the definite exceptions of $v \in \{9,12,13\}$ and possible exceptions of $v \in \{24,36,42\}$.*
(2) *There exists an optimal $(2^n,4,2)$-OOC for any integer $n \geqslant 3$.*
(3) *If there exists a $\mathrm{CSQS}(v)$, then there exists an optimal $(2^n v,4,2)$-OOC for any integer $n \geqslant 3$.*

In [4] it is pointed out that if there exists a $\mathrm{CSQS}(v)$ with $v \equiv 2,10 \pmod{12}$ and an optimal $(48,4,2)$-OOC, then there exists an optimal $(24v,4,2)$-OOC. Actually it is impossible since the following theorem.

**Theorem 6.3.** *There does not exist an optimal $(v,4,2)$-OOC for any $v \equiv 0 \pmod{24}$.*

**Proof.** For a fixed positive integer $v$, let $p(v)$ denote the largest possible number of blocks in a 3-$(v,4,1)$ packing design. By [19] it is known that if $v \equiv 0 \pmod 6$, then

$$p(v) \leqslant \left\lfloor \frac{v}{4}\left\lfloor \frac{v-1}{3}\left\lfloor \frac{v-2}{2}\right\rfloor\right\rfloor - 1\right\rfloor.$$

Let $\Phi(v)$ denote the largest possible size of a $(v,4,2)$-OOC. By Lemma 6.1, it is obvious that $\Phi(v) \leqslant p(v)/v$. In the following we always assume that $v=24r$, where $r \geqslant 1$. Then we have that $p(v) \leqslant 576r^3 - 72r^2 - 1$. Suppose that there exists an optimal $(v,4,2)$-OOC. Then by the upper-bound formula (6.1), $\Phi(v) = 24r^2 - 3r$. It follows that $v\Phi(v) > p(v)$, contradiction. The assertion then follows. $\square$

Optimal $(v,4,2)$-OOCs are also related to $\mathrm{RoSQS}(v)$'s. Bitan and Etzion have pointed out in [2] that the existence of an $\mathrm{RoSQS}(v+1)$ implies the existence of an optimal $(v,4,2)$-OOC. Analogous to Theorem 4.13, we have

**Theorem 6.4.** *(See [17].)*

(1) *There exists an optimal $(v,4,2)$-OOC for all $v \equiv 1,3 \pmod 6$ with $4 \leqslant v < 100$ except for $v \in \{9,13\}$ and possibly for $v \in \{45,55,69,81,85,91,97\}$.*
(2) *There exists an optimal $(uv,4,2)$-OOC for any $u \in \{4^n-1\colon \text{integer } n \geqslant 1\} \cup \{1,27,33,39,51,87\}$ and $v$ is an integer taken from set $P = \{p \equiv 7 \pmod{12}\colon\ p \text{ is a prime}\} \cup \{2^n-1\colon \text{odd integer } n>1\} \cup \{25,37,61,73\}$, or a product of such integers.*

In [2] Bitan and Etzion also pointed out that optimal $(v,4,2)$-OOCs are related to another special kind of $\mathrm{SQS}(v)$ called $S$-cyclic $\mathrm{SQS}(v)$. An *$S$-cyclic* $\mathrm{SQS}(v)$ is a $\mathrm{CSQS}(v)$ with each block-orbit invariant under the mapping $i \to -i \pmod v$. The full orbits of an $S$-cyclic $\mathrm{SQS}(v)$ correspond to an optimal $(v,4,2)$-OOC [2]. The necessary condition for the existence of an $S$-cyclic $\mathrm{SQS}(v)$ is $v=2n$ or $4n$, where the prime factors of $n$ are all of the form $12s+1$ or $12s+5$ $(s=0,1,2,\ldots)$ [9]. We summarize the known results partially on $S$-cyclic SQS$(v)$ as follows. For more information the reader may refer to [28].

**Theorem 6.5.** *(See [2,9].)*

(1) *There exists an $S$-cyclic* SQS(52) *and an optimal* $(52,4,2)$*-OOC.*
(2) *There exists an $S$-cyclic* SQS$(4p)$ *and an optimal* $(4p,4,2)$*-OOC for any prime* $p\equiv 5 \pmod{12}$ *and* $p<1500000$.

For obtaining some optimal OOCs via our constructions shown in Sections 2–4 we need the following theorem, which is a generalization of Theorem 5.1 in [4].

**Theorem 6.6.** *Let $g\geqslant k\geqslant 3$. Suppose that there exists a strictly cyclic* $0$-FG$(3,(\emptyset,k),gn)$ *of type $g^n$. If there exists an optimal* $(g,k,2)$*-OOC, then there exists an optimal* $(gn,k,2)$*-OOC.*

**Proof.** Let $\mathcal{F}$ be the family of base blocks of a strictly cyclic 0-FG$(3,(\emptyset,k),gn)$ of type $g^n$. Let $\mathcal{E}$ be the family of codewords of an optimal $(g,k,2)$-OOC. For each $B=\{b_0,b_1,\ldots,b_{k-1}\}\in\mathcal{E}$ we take

$$nB=\{nb_0,nb_1,\ldots,nb_{k-1}\} \pmod{gn}.$$

Then the family of codewords $\mathcal{F}\cup\{nB\colon B\in\mathcal{E}\}$ forms the desired optimal $(gn,k,2)$-OOC.

For checking optimality of the required design, it suffices to show that

$$\frac{(gn-1)(gn-2)-(g-1)(g-2)}{k(k-1)(k-2)}+\left\lfloor\frac{1}{k}\left\lfloor\frac{g-1}{k-1}\left\lfloor\frac{g-2}{k-2}\right\rfloor\right\rfloor\right\rfloor=\left\lfloor\frac{1}{k}\left\lfloor\frac{gn-1}{k-1}\left\lfloor\frac{gn-2}{k-2}\right\rfloor\right\rfloor\right\rfloor. \tag{6.3}$$

First we will prove that if $a$, $b$, $c$ are positive integers, then $\lfloor\frac{1}{a}\lfloor\frac{c}{b}\rfloor\rfloor=\lfloor\frac{c}{ab}\rfloor$. Let $c=xb+y$, $0\leqslant y\leqslant b-1$. Let $x=x_1a+y_1$, $0\leqslant y_1\leqslant a-1$. It follows that $\lfloor\frac{c}{ab}\rfloor=\lfloor\frac{x}{a}+\frac{y}{ab}\rfloor=\lfloor x_1+\frac{y_1}{a}+\frac{y}{ab}\rfloor=\lfloor x_1+\frac{y_1b+y}{ab}\rfloor=x_1=\lfloor\frac{1}{a}\lfloor\frac{c}{b}\rfloor\rfloor$.

By Lemma 2.1(3) we have that $gn-2\equiv g-2\equiv 0 \pmod{k-2}$. It is readily checked that Eq. (6.3) holds. This completes the proof. □

Note that by Theorem 6.2, there exists an optimal $(8,4,2)$-OOC. Thus inductively applying Theorem 6.6 and Lemma 4.6 we have another proof of Theorem 6.2(2).

**Lemma 6.7.** *There exists a strictly cyclic $H(7,4,4,3)$.*

**Proof.** The required design is constructed on $Z_{28}$ with groups $\{\{7i+j\colon 0\leqslant i\leqslant 3\}\colon 0\leqslant j\leqslant 6\}$. Base blocks for this design are given below. All other blocks are obtained by developing these base blocks by $+1$ modulo 28.

$\{0,1,2,4\}$, $\{0,1,5,6\}$, $\{0,1,9,10\}$, $\{0,1,11,16\}$, $\{0,1,12,25\}$, $\{0,1,13,18\}$,
$\{0,1,17,26\}$, $\{0,2,5,10\}$, $\{0,2,6,25\}$, $\{0,2,8,13\}$, $\{0,2,11,22\}$, $\{0,2,12,15\}$,
$\{0,2,17,20\}$, $\{0,2,18,24\}$, $\{0,3,6,23\}$, $\{0,3,12,22\}$, $\{0,4,8,16\}$, $\{0,4,9,17\}$,
$\{0,4,10,20\}$, $\{0,4,13,19\}$. □

**Lemma 6.8.** *There exists a strictly cyclic $H(11,4,4,3)$.*

**Proof.** The required design is constructed on $Z_{44}$ with groups $\{\{11i+j\colon 0\leqslant i\leqslant 3\}\colon 0\leqslant j\leqslant 10\}$. The 60 base blocks for this design can be obtained by multiplying each of the following 12 base blocks by $5^i$, $i=0,1,2,3,4$. All other blocks are obtained by developing these base blocks by $+1$ modulo 44.

$\{0,1,2,4\}$, $\{0,1,5,6\}$, $\{0,1,7,8\}$, $\{0,1,13,18\}$, $\{0,1,14,41\}$, $\{0,1,16,31\}$,
$\{0,1,17,29\}$, $\{0,1,21,30\}$, $\{0,1,28,42\}$, $\{0,2,8,12\}$, $\{0,2,16,29\}$, $\{0,2,20,36\}$. □

**Lemma 6.9.** *There exists an optimal* $(36, 4, 2)$*-OOC.*

**Proof.** We only list the base blocks for corresponding strictly cyclic 3-$(36, 4, 1)$ packing design over $Z_{36}$.

$\{0,1,2,4\}$, $\{0,1,5,6\}$, $\{0,1,7,8\}$, $\{0,1,9,10\}$, $\{0,1,11,12\}$, $\{0,1,13,14\}$,
$\{0,1,15,16\}$, $\{0,1,17,19\}$, $\{0,1,18,33\}$, $\{0,1,20,34\}$, $\{0,2,5,7\}$, $\{0,2,6,8\}$,
$\{0,2,9,11\}$, $\{0,2,10,12\}$, $\{0,2,13,15\}$, $\{0,2,14,18\}$, $\{0,2,16,21\}$, $\{0,2,17,24\}$,
$\{0,3,6,11\}$, $\{0,3,7,15\}$, $\{0,3,9,16\}$, $\{0,3,10,17\}$, $\{0,3,12,31\}$, $\{0,3,13,23\}$,
$\{0,3,14,27\}$, $\{0,3,18,29\}$, $\{0,3,19,28\}$, $\{0,3,20,26\}$, $\{0,3,24,32\}$, $\{0,3,25,30\}$,
$\{0,4,8,20\}$, $\{0,4,9,30\}$, $\{0,4,10,29\}$, $\{0,4,11,26\}$, $\{0,4,13,19\}$, $\{0,4,14,31\}$,
$\{0,4,15,23\}$, $\{0,4,17,25\}$, $\{0,4,18,27\}$, $\{0,5,10,28\}$, $\{0,5,12,21\}$, $\{0,5,13,29\}$,
$\{0,5,15,24\}$, $\{0,5,16,27\}$, $\{0,5,18,25\}$, $\{0,6,12,26\}$, $\{0,6,14,22\}$, $\{0,6,17,29\}$,
$\{0,6,18,28\}$. □

**Lemma 6.10.** *There exists an optimal* $(42, 4, 2)$*-OOC.*

**Proof.** We only list the base blocks for corresponding strictly cyclic 3-$(42, 4, 1)$ packing design over $Z_{42}$.

$\{0,1,2,4\}$, $\{0,1,5,6\}$, $\{0,1,7,8\}$, $\{0,1,9,10\}$, $\{0,1,11,12\}$, $\{0,1,13,14\}$,
$\{0,1,15,16\}$, $\{0,1,17,18\}$, $\{0,1,19,20\}$, $\{0,1,21,40\}$, $\{0,1,22,39\}$, $\{0,2,5,7\}$,
$\{0,2,6,8\}$, $\{0,2,9,11\}$, $\{0,2,10,12\}$, $\{0,2,13,15\}$, $\{0,2,14,16\}$, $\{0,2,17,22\}$,
$\{0,2,18,20\}$, $\{0,2,19,27\}$, $\{0,2,21,25\}$, $\{0,3,6,10\}$, $\{0,3,8,11\}$, $\{0,3,9,12\}$,
$\{0,3,13,16\}$, $\{0,3,14,17\}$, $\{0,3,15,18\}$, $\{0,3,19,24\}$, $\{0,3,20,26\}$, $\{0,3,21,35\}$,
$\{0,3,22,38\}$, $\{0,4,8,13\}$, $\{0,4,10,16\}$, $\{0,4,11,22\}$, $\{0,4,12,30\}$, $\{0,4,14,19\}$,
$\{0,4,15,31\}$, $\{0,4,17,35\}$, $\{0,4,18,24\}$, $\{0,4,20,32\}$, $\{0,4,27,33\}$, $\{0,4,28,37\}$,
$\{0,4,29,34\}$, $\{0,5,10,24\}$, $\{0,5,11,21\}$, $\{0,5,12,36\}$, $\{0,5,14,29\}$, $\{0,5,15,22\}$
$\{0,5,16,35\}$, $\{0,5,18,31\}$, $\{0,5,20,30\}$, $\{0,6,13,35\}$, $\{0,6,14,26\}$, $\{0,6,17,31\}$,
$\{0,6,20,33\}$, $\{0,6,21,29\}$, $\{0,6,23,30\}$, $\{0,6,27,34\}$, $\{0,7,16,33\}$, $\{0,7,17,30\}$,
$\{0,7,22,34\}$, $\{0,8,16,32\}$, $\{0,8,17,29\}$, $\{0,8,18,33\}$, $\{0,8,19,28\}$, $\{0,8,22,31\}$,
$\{0,9,19,30\}$, $\{0,9,22,32\}$. □

**Construction 6.11.**

(1) *Let* $v \equiv 0 \pmod 2$. *Suppose that there exists an optimal* $(v, 4, 2)$*-OOC. Then there exists an optimal* $(5^a v, 4, 2)$*-OOC for any integer* $a \geqslant 0$.
(2) *Let* $v \equiv 0 \pmod 8$. *Suppose that there exists an optimal* $(v, 4, 2)$*-OOC. Then there exists an optimal* $(2^a 5^b v, 4, 2)$*-OOC for any integers* $a, b \geqslant 0$.
(3) *Let* $v \equiv 12 \pmod{24}$. *Suppose that there exists an optimal* $(v, 4, 2)$*-OOC. Then there exists an optimal* $(3^a 5^b v, 4, 2)$*-OOC for any integers* $a, b \geqslant 0$.
(4) *Let* $v \equiv 12 \pmod{24}$. *Suppose that there exist an* RoSQS$(u+1)$ *and an optimal* $(v, 4, 2)$*-OOC. Then there exists an optimal* $(3^a 5^b u^c v, 4, 2)$*-OOC for any integers* $a, b, c \geqslant 0$.
(5) *Let* $g \equiv 0 \pmod 8$. *Suppose that there exist a strictly cyclic* $H(n, g, 4, 3)$ *and an optimal* $(g, 4, 2)$*-OOC. Then there exists an optimal* $(2^a 5^b n^c g, 4, 2)$*-OOC for any integers* $a, b, c \geqslant 0$.

**Proof.** (1) Use induction on $a$. The case of $a = 0$ is given by the assumption. Suppose that there exists an optimal $(5^r v, 4, 2)$-OOC for $0 \leqslant r \leqslant a$. Since $v \equiv 0 \pmod 2$, by Lemma 4.9 we have a strictly cyclic 0-FG$(3, (\emptyset, 4), 5^{a+1} v)$ of type $(5^a v)^5$. Then apply Theorem 6.6 to obtain an optimal $(5^{a+1} v, 4, 2)$-OOC.

(2) If $v \equiv 0 \pmod 8$, by Lemma 4.7 we have a strictly cyclic 0-FG$(3, (\emptyset, 4), 2^{a+1} v)$ of type $(2^a v)^2$ for any integer $a \geqslant 0$. Then similar arguments as in (1), using induction on $a$, we have an optimal $(2^a v, 4, 2)$-OOC for any integer $a \geqslant 0$. Thus by the results from (1), an optimal $(2^a 5^b v, 4, 2)$-OOC for any integers $a, b \geqslant 0$ is obtained.

(3) If $v \equiv 12 \pmod{24}$, by Lemma 4.8 we have a strictly cyclic 0-FG$(3, (\emptyset, 4), 3^{a+1}v)$ of type $(3^a v)^3$ for any integer $a \geqslant 0$. Then similar arguments as in (1), using induction on $a$, we have an optimal $(3^a v, 4, 2)$-OOC for any integer $a \geqslant 0$. Thus by the results from (1), an optimal $(3^a 5^b v, 4, 2)$-OOC for any integers $a, b \geqslant 0$ is obtained.

(4) If $v \equiv 12 \pmod{24}$ and $u \equiv 1 \pmod 6$, by Lemma 4.14 with the given RoSQS$(u+1)$ there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), uv)$ of type $v^u$. Then apply Theorem 6.6 with the given optimal $(v, 4, 2)$-OOC to obtain an optimal $(uv, 4, 2)$-OOC. If $u \equiv 3 \pmod 6$, by Lemma 4.14 with the given RoSQS$(u+1)$ there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), uv)$ of type $(3v)^{u/3}$. By the results from (3), there exists an optimal $(3v, 4, 2)$-OOC. Then apply Theorem 6.6 to obtain an optimal $(uv, 4, 2)$-OOC. Using induction on $c$ and repeating the above process, an optimal $(u^c v, 4, 2)$-OOC for any integer $c \geqslant 0$ is obtained. Thus by the results from (3), we have an optimal $(3^a 5^b u^c v, 4, 2)$-OOC for any integers $a, b, c \geqslant 0$.

(5) If $g \equiv 0 \pmod 8$ and $n \equiv 1 \pmod 2$, by Lemma 4.11 with the given strictly cyclic $H(n, g, 4, 3)$ there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), gn)$ of type $g^n$. Then apply Theorem 6.6 with the given optimal $(g, 4, 2)$-OOC to obtain an optimal $(gn, 4, 2)$-OOC. If $n \equiv 0 \pmod 2$, by Lemma 4.11 with the given strictly cyclic $H(n, g, 4, 3)$ there exists a strictly cyclic 0-FG$(3, (\emptyset, 4), gn)$ of type $(2g)^{n/2}$. By the results from (2), there exists an optimal $(2g, 4, 2)$-OOC. Then apply Theorem 6.6 to obtain an optimal $(gn, 4, 2)$-OOC. By Corollary 3.10, if there exists a strictly cyclic $H(n, g, 4, 3)$, there exists a strictly cyclic $H(n, gn^c, 4, 3)$ for any integer $c \geqslant 0$. Using induction on $c$ and repeating the above process, an optimal $(n^c g, 4, 2)$-OOC for any integer $c \geqslant 0$ is obtained. Thus by the results from (2), we have an optimal $(2^a 5^b n^c g, 4, 2)$-OOC for any integers $a, b, c \geqslant 0$. □

By Construction 6.11, if one has proper $(v, 4, 2)$-OOCs, some infinite families of OOCs will be obtained. As examples, we have the following corollaries.

**Corollary 6.12.** *There exists an optimal* $(5^a v, 4, 2)$*-OOC for any integer* $a \geqslant 0$ *and* $v \in \{14, 18, 20, 42\}$.

**Proof.** Apply Construction 6.11(1) with an optimal $(v, 4, 2)$-OOC for $v \in \{14, 18, 20, 42\}$ from Theorem 6.2 and Lemma 6.10 to obtain the required designs. □

**Corollary 6.13.** *Suppose that there exists a* CSQS$(v)$. *Then there exists an optimal* $(2^a 5^b v, 4, 2)$*-OOC for any integers* $a \geqslant 3$ *and* $b \geqslant 0$.

**Proof.** By Theorem 6.2, if there exists a CSQS$(v)$, there exists an optimal $(2^a v, 4, 2)$-OOC for any integer $a \geqslant 3$. Then apply Construction 6.11(1) to complete the proof. □

**Corollary 6.14.** *Suppose that there exists an* RoSQS$(u+1)$. *Then there exists an optimal* $(3^a 5^b u^c 36, 4, 2)$*-OOC for any integers* $a, b, c \geqslant 0$.

**Proof.** Apply Construction 6.11(4) with an optimal $(36, 4, 2)$-OOC from Lemma 6.9 to obtain the required designs. □

**Corollary 6.15.** *Suppose that there exists a* CSQS$(n)$ *for* $n \equiv 2, 10 \pmod{12}$ *and an optimal* $(g, 4, 2)$*-OOC for* $g \equiv 0 \pmod 8$. *Then there exists an optimal* $(2^a 5^b n^c g, 4, 2)$*-OOC for any integers* $a, b, c \geqslant 0$.

**Proof.** By Lemma 3.12, if there exists a CSQS$(n)$ for $n \equiv 2, 10 \pmod{12}$, then for any $g \equiv 0 \pmod 8$, there exists a strictly cyclic $H(n, g, 4, 3)$. Apply Construction 6.11(5) with the given optimal $(g, 4, 2)$-OOC to complete the proof. □

**Corollary 6.16.** *There exist an optimal* $(2^{a+3} 5^b 7^c, 4, 2)$*-OOC and an optimal* $(2^{a+3} 5^b 11^c, 4, 2)$*-OOC for any integers* $a, b, c \geqslant 0$.

**Proof.** Start with a strictly cyclic $H(n,4,4,3)$ for $n \in \{7,11\}$ from Lemmas 6.7 and 6.8. Apply Corollary 3.10 to obtain a strictly cyclic $H(n,8,4,3)$. Then by Construction 6.11(5) with an optimal $(8,4,2)$-OOC from Theorem 6.2 to complete the proof. □

**Theorem 6.17.** *Let $n$ be a positive integer. If $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where $p_i = 13$ or $p_i$ is a prime, $p_i \equiv 5 \pmod{12}$ and $p_i < 1500000$, $r_i \geqslant 1$ for $1 \leqslant i \leqslant s$, there are a cyclic 0-FG*$(3,(\emptyset,4),4n)$ of type $4^n$, a CSQS$(4n)$ and an optimal $(4n,4,2)$-OOC.*

**Proof.** Let $d(n) = r_1 + \cdots + r_s$. Use induction on $d(n)$. If $d(n) = 1$, then $s = 1$ and $r_1 = 1$. By Theorem 6.5 there exist an $S$-cyclic SQS$(4p_1)$ and an optimal $(4p_1,4,2)$-OOC. Since no $S$-cyclic SQS can contain a half-orbit and any $S$-cyclic SQS$(v)$ for $v \equiv 0 \pmod 4$ must contain a quarter-orbit [9], there exists a cyclic 0-FG*$(3,(\emptyset,4),4p_1)$ of type $4^{p_1}$.

Assume that the result holds for $d(n) = r$. When $d(n) = r+1$, let $n = n'p'$, where $d(n') = r$, $p' \equiv 5 \pmod{12}$ and $p' < 1500000$, or $p' = 13$. Start with a cyclic 0-FG*$(3,(\emptyset,4),4n')$ of type $4^{n'}$, which exists by assumption. Apply Construction 4.1 to obtain a cyclic 0-FG*$(3,(\emptyset,4),4p'n')$ of type $(4p')^{n'}$. It is easy to verify that the base blocks in the cyclic 0-FG*$(3,(\emptyset,4),4p'n')$ of type $(4p')^{n'}$, together with the base blocks of a cyclic 0-FG*$(3,(\emptyset,4),4p')$ of type $4^{p'}$, form the set of base blocks of a cyclic 0-FG*$(3,(\emptyset,4),4n)$ of type $4^n$. Obviously it is also an optimal $(4n,4,2)$-OOC. Apply Construction 2.2 with a CSQS(4) from Theorem 1.1 to obtain a CSQS$(4n)$. This completes the proof. □

To conclude, we give some analysis on OOCs with orders $v \equiv 0 \pmod{24}$. As pointed out in Theorem 6.3, there does not exist an optimal $(v,4,2)$-OOC for any $v \equiv 0 \pmod{24}$. However observing the proof of Theorem 6.3 and the relation between OOCs and strictly cyclic packing designs from Lemma 6.1 we may discuss the existence of maximal strictly cyclic packing design.

A strictly cyclic $t$-$(v,k,1)$ packing design is said to be *maximal* if it contains the largest possible blocks. In the case of $k = 4$ and $t = 3$, such a design will be denoted by sMPQS$(v)$. For more information on MPQS, the reader may refer to [14]. Thus by the definition of sMPQS and the proof of Theorem 6.3, the number of the largest blocks on sMPQS$(v)$, $v \equiv 0 \pmod{24}$, is

$$\left\lfloor \frac{1}{4} \left\lfloor \frac{v-1}{3} \left\lfloor \frac{v-2}{2} \right\rfloor \right\rfloor \right\rfloor - 1. \tag{6.4}$$

It follows that according to Chu and Colbourn's result [3] we have

**Lemma 6.18.** *(See [3].) There exists an* sMPQS(24).

Similar arguments as in Theorem 6.6 and Construction 6.11, the following results should be clear.

**Theorem 6.19.** *Suppose that there exists a strictly cyclic 0-FG$(3,(\emptyset,4),gn)$ of type $g^n$. If there exists an* sMPQS$(g)$, *then there exists an* sMPQS$(gn)$.

**Construction 6.20.**

(1) *Let $v \equiv 0 \pmod{24}$. Suppose that there exists an* sMPQS$(v)$. *Then there exists an* sMPQS$(2^a3^b5^cv)$ *for any integers $a,b,c \geqslant 0$.*
(2) *Let $v \equiv 0 \pmod{24}$. Suppose that there exist an* RoSQS$(u+1)$ *and an* sMPQS$(v)$. *Then there exists an* sMPQS$(2^a3^b5^cu^dv,4,2)$ *for any integers $a,b,c,d \geqslant 0$.*
(3) *Let $g \equiv 0 \pmod{24}$. Suppose that there exist a strictly cyclic $H(n,g,4,3)$ and an* sMPQS$(g)$. *Then there exists an* sMPQS$(2^a3^b5^cn^dg)$ *for any integers $a,b,c,d \geqslant 0$.*

**Corollary 6.21.** *Suppose that there exist a* CSQS($n$) *for* $n \equiv 2, 10 \pmod{12}$ *and an* sMPQS($g$) *for* $g \equiv 0 \pmod{24}$. *Then there exists an* sMPQS($2^a 3^b 5^c n^d g$) *for any integers* $a, b, c, d \geqslant 0$.

**Proof.** By Lemma 3.12, if there exists a CSQS($n$) for $n \equiv 2, 10 \pmod{12}$, then there exists a strictly cyclic $H(n, g, 4, 3)$ for $g \equiv 0 \pmod{24}$. Apply Construction 6.20(3) with the given sMPQS($g$) to obtain the required designs. $\square$

**Corollary 6.22.** *There exist an* sMPQS($2^a 3^b 5^c 7^d \cdot 24$) *and an* sMPQS($2^a 3^b 5^c 11^d \cdot 24$) *for any integers* $a, b, c, d \geqslant 0$.

**Proof.** Start with a strictly cyclic $H(n, 4, 4, 3)$ for $n \in \{7, 11\}$ from Lemmas 6.7 and 6.8. Apply Corollary 3.10 to obtain a strictly cyclic $H(n, 24, 4, 3)$. Then apply Construction 6.20(3) with an sMPQS(24) from Lemma 6.18 to complete the proof. $\square$

## 7. More optimal OOCs

In this section some new optimal $(v, 4, 2)$-OOCs are given by direct construction for $v \equiv 2, 10 \pmod{12}$. We only list the base blocks for corresponding optimal strictly cyclic 3-$(v, 4, 1)$ packing designs over $Z_v$. Combining these new optimal $(v, 4, 2)$-OOCs and Construction 6.11, one can have some infinite families of optimal OOCs.

For shorting the list of base blocks for these designs, all base blocks are divided into three parts. The first part consists of $\{0, i, 2i, \frac{v}{2} + i\}$, $1 \leqslant i \leqslant \frac{v-2}{4}$, in which if $v = 54$, $i \neq 9$ and if $v = 78$, $i \neq 13$. The second part is $\{0, 2i-1, 2i+2j-1, 4i+2j-2\}$, $1 \leqslant i \leqslant \frac{v-6}{4}$, $1 \leqslant j \leqslant \frac{v-2}{4} - i$. The third part is displayed below.

**Example 7.1.** There exists an optimal (46, 4, 2)-OOC.

$\{0, 2, 6, 8\}$, $\{0, 2, 10, 12\}$, $\{0, 2, 14, 16\}$, $\{0, 2, 18, 20\}$, $\{0, 4, 10, 14\}$, $\{0, 4, 12, 16\}$,
$\{0, 4, 18, 22\}$, $\{0, 6, 14, 20\}$, $\{0, 6, 16, 22\}$, $\{0, 6, 18, 24\}$, $\{0, 8, 18, 26\}$, $\{0, 8, 22, 30\}$,
$\{0, 10, 22, 32\}$, $\{0, 12, 20, 32\}$, $\{0, 16, 20, 36\}$, $\{0, 20, 22, 42\}$.

**Example 7.2.** There exists an optimal (54, 4, 2)-OOC.

$\{0, 2, 6, 8\}$, $\{0, 2, 10, 12\}$, $\{0, 2, 14, 16\}$, $\{0, 2, 18, 20\}$, $\{0, 2, 22, 24\}$, $\{0, 2, 26, 30\}$,
$\{0, 4, 10, 14\}$, $\{0, 4, 12, 16\}$, $\{0, 4, 18, 22\}$, $\{0, 4, 20, 26\}$, $\{0, 4, 24, 34\}$, $\{0, 4, 32, 38\}$,
$\{0, 6, 14, 20\}$, $\{0, 6, 16, 44\}$, $\{0, 6, 18, 28\}$, $\{0, 6, 24, 36\}$, $\{0, 6, 32, 42\}$, $\{0, 8, 18, 32\}$,
$\{0, 8, 20, 36\}$, $\{0, 8, 22, 34\}$, $\{0, 8, 24, 38\}$, $\{0, 8, 26, 42\}$, $\{0, 8, 28, 40\}$, $\{0, 8, 30, 44\}$.

**Example 7.3.** There exists an optimal (62, 4, 2)-OOC.

$\{0, 2, 6, 8\}$, $\{0, 2, 12, 14\}$, $\{0, 2, 16, 18\}$, $\{0, 2, 22, 24\}$, $\{0, 2, 28, 30\}$, $\{0, 4, 12, 16\}$,
$\{0, 4, 18, 22\}$, $\{0, 4, 20, 24\}$, $\{0, 4, 26, 30\}$, $\{0, 4, 28, 32\}$, $\{0, 6, 10, 16\}$, $\{0, 6, 18, 24\}$,
$\{0, 6, 20, 26\}$, $\{0, 6, 22, 28\}$, $\{0, 6, 30, 36\}$, $\{0, 8, 10, 18\}$, $\{0, 8, 14, 22\}$, $\{0, 8, 26, 34\}$,
$\{0, 8, 30, 38\}$, $\{0, 10, 14, 24\}$, $\{0, 10, 28, 38\}$, $\{0, 10, 30, 40\}$, $\{0, 12, 22, 34\}$, $\{0, 12, 30, 42\}$,
$\{0, 14, 26, 40\}$, $\{0, 14, 30, 44\}$, $\{0, 16, 24, 40\}$, $\{0, 16, 26, 42\}$, $\{0, 16, 28, 44\}$, $\{0, 18, 20, 38\}$,
$\{0, 20, 28, 48\}$, $\{0, 24, 26, 50\}$.

**Example 7.4.** There exists an optimal $(78, 4, 2)$-OOC.

$\{0,2,6,8\},\quad \{0,2,10,12\},\quad \{0,2,14,16\},\quad \{0,2,18,20\},\quad \{0,2,22,24\},\quad \{0,2,26,28\},$
$\{0,2,30,32\},\quad \{0,2,34,36\},\quad \{0,2,38,42\},\quad \{0,4,10,14\},\quad \{0,4,12,16\},\quad \{0,4,18,22\},$
$\{0,4,20,24\},\quad \{0,4,26,30\},\quad \{0,4,28,32\},\quad \{0,4,34,38\},\quad \{0,4,36,46\},\quad \{0,6,14,20\},$
$\{0,6,16,22\},\quad \{0,6,18,24\},\quad \{0,6,26,32\},\quad \{0,6,28,34\},\quad \{0,6,30,36\},\quad \{0,6,38,44\},$
$\{0,8,18,26\},\quad \{0,8,20,28\},\quad \{0,8,22,30\},\quad \{0,8,24,32\},\quad \{0,8,34,42\},\quad \{0,8,36,50\},$
$\{0,8,38,46\},\quad \{0,10,22,32\},\quad \{0,10,24,34\},\quad \{0,10,26,48\},\quad \{0,10,28,60\},\quad \{0,10,30,58\},$
$\{0,10,36,52\},\quad \{0,10,38,50\},\quad \{0,10,40,62\},\quad \{0,12,26,60\},\quad \{0,12,28,52\},\quad \{0,12,30,64\},$
$\{0,12,32,58\},\quad \{0,12,34,46\},\quad \{0,12,36,48\},\quad \{0,12,38,62\},\quad \{0,14,30,62\},\quad \{0,14,32,56\},$
$\{0,14,34,58\},\quad \{0,14,36,60\},\quad \{0,14,38,52\},\quad \{0,16,34,50\},\quad \{0,16,36,58\},\quad \{0,18,38,56\}.$

**Example 7.5.** There exists an optimal $(86, 4, 2)$-OOC.

$\{0,2,6,8\},\quad \{0,2,10,12\},\quad \{0,2,14,16\},\quad \{0,2,18,20\},\quad \{0,2,22,24\},\quad \{0,2,26,28\},$
$\{0,2,30,32\},\quad \{0,2,34,36\},\quad \{0,2,38,40\},\quad \{0,2,42,46\},\quad \{0,4,10,14\},\quad \{0,4,12,16\},$
$\{0,4,18,22\},\quad \{0,4,20,24\},\quad \{0,4,26,30\},\quad \{0,4,28,32\},\quad \{0,4,34,38\},\quad \{0,4,36,40\},$
$\{0,4,42,48\},\quad \{0,6,14,20\},\quad \{0,6,16,22\},\quad \{0,6,18,24\},\quad \{0,6,26,32\},\quad \{0,6,28,34\},$
$\{0,6,30,36\},\quad \{0,6,38,50\},\quad \{0,6,40,52\},\quad \{0,6,42,54\},\quad \{0,8,18,26\},\quad \{0,8,20,28\},$
$\{0,8,22,30\},\quad \{0,8,24,32\},\quad \{0,8,34,42\},\quad \{0,8,36,44\},\quad \{0,8,38,46\},\quad \{0,8,40,54\},$
$\{0,10,22,32\},\quad \{0,10,24,34\},\quad \{0,10,26,36\},\quad \{0,10,28,38\},\quad \{0,10,30,66\},\quad \{0,10,40,50\},$
$\{0,10,42,52\},\quad \{0,12,26,42\},\quad \{0,12,28,70\},\quad \{0,12,30,68\},\quad \{0,12,32,66\},\quad \{0,12,34,62\},$
$\{0,12,36,64\},\quad \{0,12,38,58\},\quad \{0,12,40,60\},\quad \{0,12,56,72\},\quad \{0,14,32,58\},\quad \{0,14,34,56\},$
$\{0,14,36,60\},\quad \{0,14,38,52\},\quad \{0,14,40,64\},\quad \{0,14,42,68\},\quad \{0,14,44,66\},\quad \{0,16,34,50\},$
$\{0,16,36,54\},\quad \{0,16,38,64\},\quad \{0,16,40,56\},\quad \{0,16,48,66\},\quad \{0,18,40,58\},\quad \{0,18,42,62\}.$

**Example 7.6.** There exists an optimal $(94, 4, 2)$-OOC.

$\{0,2,6,8\},\quad \{0,2,10,12\},\quad \{0,2,14,16\},\quad \{0,2,18,20\},\quad \{0,2,22,24\},\quad \{0,2,26,28\},$
$\{0,2,30,32\},\quad \{0,2,34,36\},\quad \{0,2,38,40\},\quad \{0,2,42,44\},\quad \{0,2,46,50\},\quad \{0,4,10,14\},$
$\{0,4,12,16\},\quad \{0,4,18,22\},\quad \{0,4,20,24\},\quad \{0,4,26,30\},\quad \{0,4,28,32\},\quad \{0,4,34,38\},$
$\{0,4,36,40\},\quad \{0,4,42,46\},\quad \{0,4,44,54\},\quad \{0,6,14,20\},\quad \{0,6,16,22\},\quad \{0,6,18,24\},$
$\{0,6,26,32\},\quad \{0,6,28,34\},\quad \{0,6,30,36\},\quad \{0,6,38,44\},\quad \{0,6,40,46\},\quad \{0,6,42,48\},$
$\{0,8,18,26\},\quad \{0,8,20,28\},\quad \{0,8,22,30\},\quad \{0,8,24,32\},\quad \{0,8,34,42\},\quad \{0,8,36,44\},$
$\{0,8,38,50\},\quad \{0,8,40,48\},\quad \{0,8,46,56\},\quad \{0,8,52,64\},\quad \{0,10,22,32\},\quad \{0,10,24,34\},$
$\{0,10,26,36\},\quad \{0,10,28,38\},\quad \{0,10,30,40\},\quad \{0,10,42,58\},\quad \{0,10,44,60\},\quad \{0,10,46,62\},$
$\{0,12,26,38\},\quad \{0,12,28,40\},\quad \{0,12,30,52\},\quad \{0,12,32,58\},\quad \{0,12,34,46\},\quad \{0,12,36,70\},$
$\{0,12,44,62\},\quad \{0,12,48,74\},\quad \{0,12,54,76\},\quad \{0,14,30,56\},\quad \{0,14,32,72\},\quad \{0,14,34,62\},$
$\{0,14,36,76\},\quad \{0,14,38,70\},\quad \{0,14,40,68\},\quad \{0,14,42,60\},\quad \{0,14,44,58\},\quad \{0,14,46,74\},$
$\{0,14,48,66\},\quad \{0,14,52,78\},\quad \{0,16,34,54\},\quad \{0,16,36,74\},\quad \{0,16,38,72\},\quad \{0,16,40,70\},$
$\{0,16,44,66\},\quad \{0,16,46,64\},\quad \{0,16,56,76\},\quad \{0,18,42,70\},\quad \{0,18,44,68\},\quad \{0,20,42,62\},$
$\{0,20,44,64\},\quad \{0,22,46,68\}.$

**Example 7.7.** There exists an sSQS(98), i.e., an optimal $(98, 4, 2)$-OOC.

**Table 1**
Small orders for optimal $(n, 4, 2)$-OOCs for $7 \leqslant n \leqslant 100$

| $n$ | Existence | Source | $n$ | Existence | Source |
|---|---|---|---|---|---|
| 7–8 | Yes | Theorem 6.2(1) | 9 | No | Theorem 6.2(1) |
| 10–11 | Yes | Theorem 6.2(1) | 12–13 | No | Theorem 6.2(1) |
| 14–23 | Yes | Theorem 6.2(1) | 24 | No | Theorem 6.3 |
| 25–35 | Yes | Theorem 6.2(1) | 36 | Yes | Lemma 6.9 |
| 37–41 | Yes | Theorem 6.2(1) | 42 | Yes | Lemma 6.10 |
| 43–44 | Yes | Theorem 6.2(1) | 45 | ?? | |
| 46 | Yes | Example 7.1 | 47 | ?? | |
| 48 | No | Theorem 6.3 | 49 | Yes | Theorem 6.4(1) |
| 50 | Yes | Theorem 5.6 | 51 | Yes | Theorem 6.4(1) |
| 52 | Yes | Theorem 6.5(1) | 53 | ?? | |
| 54 | Yes | Example 7.2 | 55 | ?? | |
| 56 | Yes | Corollary 6.16 | 57 | Yes | Theorem 6.4(1) |
| 58 | Yes | Theorem 5.6 | 59–60 | ?? | |
| 61 | Yes | Theorem 6.4(1) | 62 | Yes | Example 7.3 |
| 63 | Yes | Theorem 6.4(2) | 64 | Yes | Theorem 6.2(2) |
| 65–66 | ?? | | 67 | Yes | Theorem 6.4(1) |
| 68 | Yes | Theorem 6.5(2) | 69 | ?? | |
| 70 | Yes | Corollary 6.12 | 71 | ?? | |
| 72 | No | Theorem 6.3 | 73 | Yes | Theorem 6.4(1) |
| 74 | Yes | Theorem 5.6 | 75 | Yes | Theorem 6.4(1) |
| 76–77 | ?? | | 78 | Yes | Example 7.4 |
| 79 | Yes | Theorem 6.4(1) | 80 | Yes | Theorems 5.6, 6.2(3) |
| 81 | ?? | | 82 | Yes | Theorem 5.6 |
| 83–85 | ?? | | 86 | Yes | Example 7.5 |
| 87 | Yes | Theorem 6.4(1) | 88 | Yes | Corollary 6.16 |
| 89 | ?? | | 90 | Yes | Corollary 6.12 |
| 91–92 | ?? | | 93 | Yes | Theorem 6.4(1) |
| 94 | Yes | Example 7.6 | 95 | ?? | |
| 96 | No | Theorem 6.3 | 97 | ?? | |
| 98 | Yes | Example 7.7 | 99 | ?? | |
| 100 | Yes | Corollary 6.12 | | | |

$$
\begin{array}{llllll}
\{0,2,6,8\}, & \{0,2,10,12\}, & \{0,2,14,16\}, & \{0,2,18,20\}, & \{0,2,22,24\}, & \{0,2,26,28\}, \\
\{0,2,30,32\}, & \{0,2,34,36\}, & \{0,2,38,40\}, & \{0,2,42,44\}, & \{0,2,46,48\}, & \{0,4,10,14\}, \\
\{0,4,12,16\}, & \{0,4,18,22\}, & \{0,4,20,24\}, & \{0,4,26,30\}, & \{0,4,28,32\}, & \{0,4,34,38\}, \\
\{0,4,36,40\}, & \{0,4,42,46\}, & \{0,4,44,52\}, & \{0,4,48,54\}, & \{0,4,50,58\}, & \{0,6,14,20\}, \\
\{0,6,16,22\}, & \{0,6,18,24\}, & \{0,6,26,32\}, & \{0,6,28,34\}, & \{0,6,30,36\}, & \{0,6,38,44\}, \\
\{0,6,40,46\}, & \{0,6,42,48\}, & \{0,8,18,26\}, & \{0,8,20,28\}, & \{0,8,22,30\}, & \{0,8,24,32\}, \\
\{0,8,34,42\}, & \{0,8,36,44\}, & \{0,8,38,46\}, & \{0,8,40,50\}, & \{0,8,56,66\}, & \{0,10,22,32\}, \\
\{0,10,24,34\}, & \{0,10,26,36\}, & \{0,10,28,38\}, & \{0,10,30,40\}, & \{0,10,44,56\}, & \{0,10,46,60\}, \\
\{0,10,48,62\}, & \{0,10,52,64\}, & \{0,12,26,38\}, & \{0,12,28,40\}, & \{0,12,30,42\}, & \{0,12,32,58\}, \\
\{0,12,34,62\}, & \{0,12,36,74\}, & \{0,12,44,60\}, & \{0,12,48,76\}, & \{0,12,50,66\}, & \{0,12,52,78\}, \\
\{0,14,30,66\}, & \{0,14,32,54\}, & \{0,14,34,74\}, & \{0,14,36,76\}, & \{0,14,38,78\}, & \{0,14,40,70\}, \\
\{0,14,42,72\}, & \{0,14,44,68\}, & \{0,14,46,82\}, & \{0,14,48,64\}, & \{0,14,58,80\}, & \{0,16,34,80\}, \\
\{0,16,36,78\}, & \{0,16,38,72\}, & \{0,16,40,56\}, & \{0,16,42,76\}, & \{0,16,44,70\}, & \{0,16,46,68\}, \\
\{0,18,38,66\}, & \{0,18,42,60\}, & \{0,18,44,62\}, & \{0,18,46,70\}, & \{0,18,48,68\}, & \{0,18,50,78\}, \\
\{0,20,42,66\}, & \{0,20,44,64\}, & \{0,20,52,76\}, & \{0,22,48,72\}. & &
\end{array}
$$

With the above direct constructions and the known results in Section 6 on optimal OOCs, we summarize the known and unknown small orders below 100 on optimal $(n, 4, 2)$-OOCs in Table 1, where the question marks "??" indicates the orders for which the existence of an optimal $(n, 4, 2)$-OOC is still undecided. Therefore, we have the following:

**Theorem 7.8.** *There exists an optimal $(n, 4, 2)$-OOC for all $7 \leqslant n \leqslant 100$ with the definite exceptions of $n \in \{9, 12, 13, 24, 48, 72, 96\}$ and possible exceptions of $n \in \{45, 47, 53, 55, 59, 60, 65, 66, 69, 71, 76, 77, 81, 83, 84, 85, 89, 91, 92, 95, 97, 99\}$.*

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